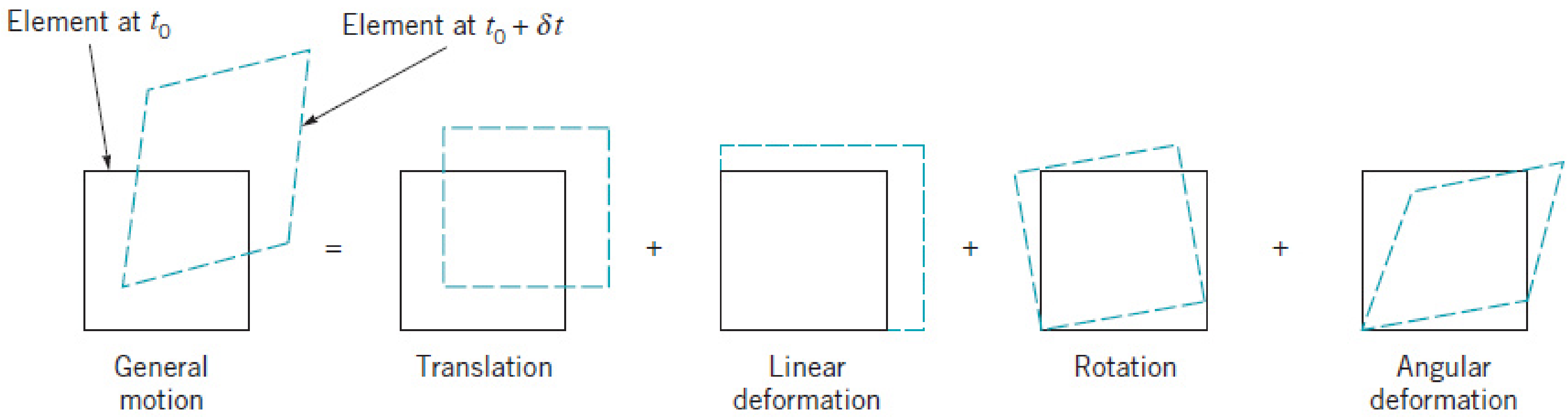
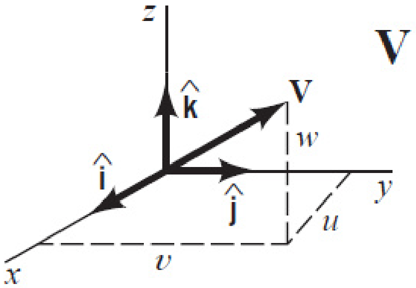
تحلیل حرکت سیال با استفاده از روش دیفرانسیلی





$$\mathbf{V} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}$$

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$$

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt}$$

$$\begin{cases} a_x = \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} \\ a_y = \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} \\ a_z = \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z} \end{cases}$$

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y} + w\frac{\partial(\ )}{\partial z}$$

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + (\mathbf{V}\cdot\nabla)(\ )$$

gradient operator, $\nabla(\ )$

$$\nabla(\ ) = \frac{\partial(\ )}{\partial x}\hat{\mathbf{i}} + \frac{\partial(\ )}{\partial y}\hat{\mathbf{j}} + \frac{\partial(\ )}{\partial z}\hat{\mathbf{k}}$$

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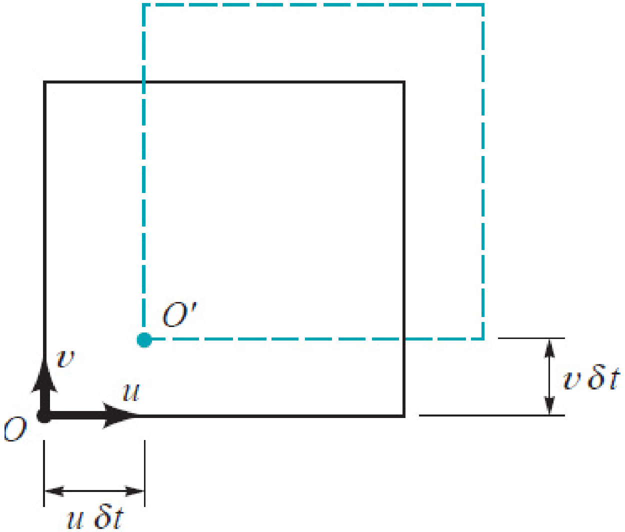
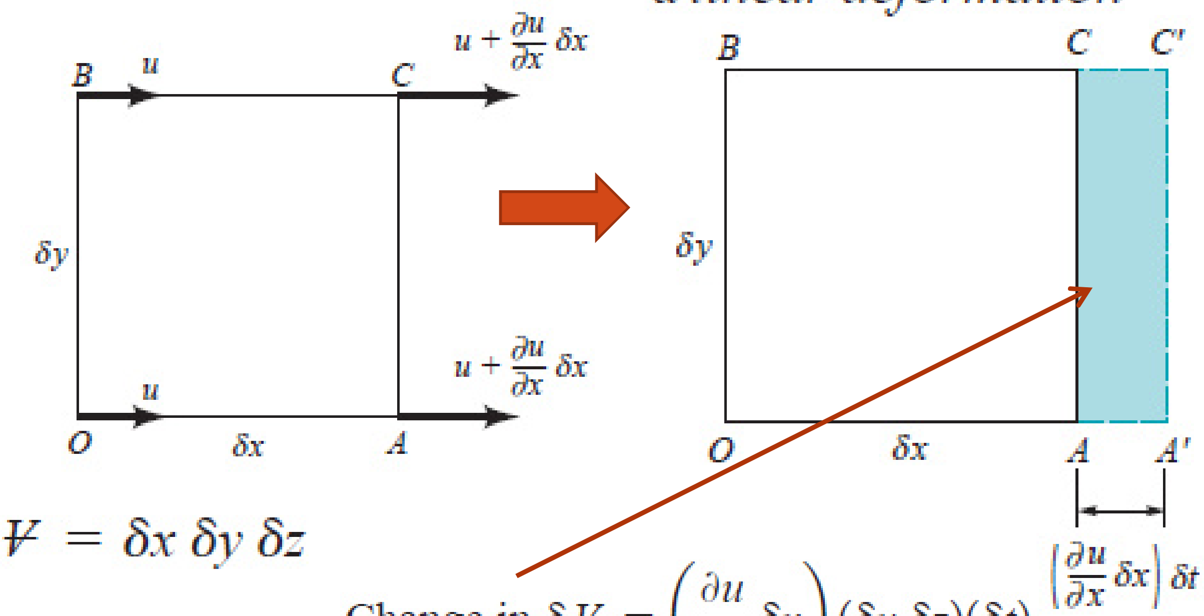
Translation of a fluid element



a *linear deformation*



$$\delta V = \delta x \, \delta y \, \delta z$$

$$\text{Change in } \delta V = \left(\frac{\partial u}{\partial x}\delta x\right)(\delta y \, \delta z)(\delta t)$$

$$\frac{1}{\delta V}\frac{d(\delta V)}{dt} = \lim_{\delta t \to 0}\left[\frac{(\partial u/\partial x)\,\delta t}{\delta t}\right] = \frac{\partial u}{\partial x}$$

If velocity gradients $\partial v/\partial y$ and $\partial w/\partial z$ are also present.

$$\frac{1}{\delta V}\frac{d(\delta V)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{V}$$

This rate of change of the volume per unit volume

for an *incompressible fluid* $\nabla \cdot \mathbf{V} = 0$

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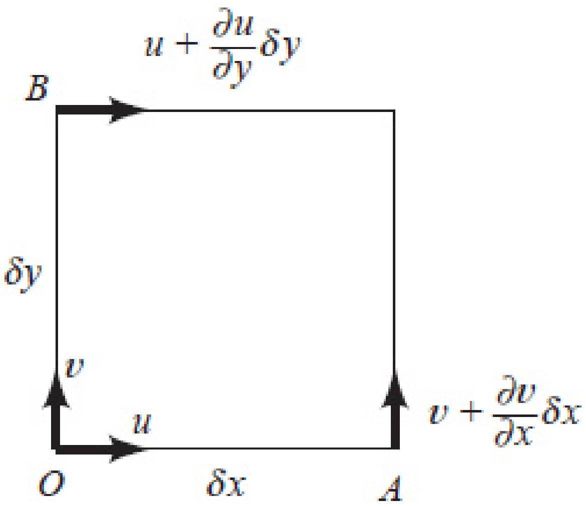
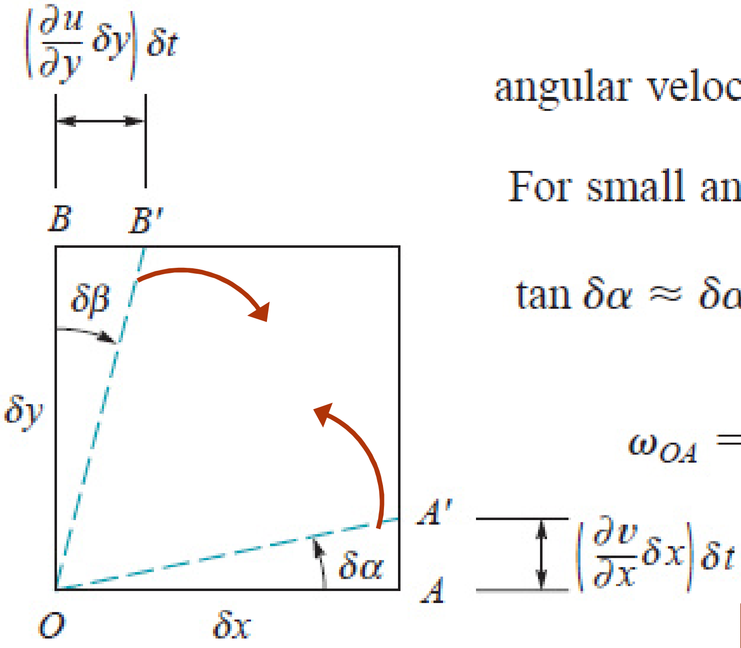
Angular Motion and Deformation





angular velocity $\omega_{OA} = \lim_{\delta t\to 0}\frac{\delta\alpha}{\delta t}$

For small angles

$$\tan\delta\alpha \approx \delta\alpha = \frac{(\partial v/\partial x)\,\delta x\,\delta t}{\delta x} = \frac{\partial v}{\partial x}\delta t$$

$$\omega_{OA} = \lim_{\delta t\to 0}\left[\frac{(\partial v/\partial x)\,\delta t}{\delta t}\right] = \frac{\partial v}{\partial x}$$

$$\omega_{OB} = \lim_{\delta t\to 0}\frac{\delta\beta}{\delta t}$$

$$\tan\delta\beta \approx \delta\beta = \frac{(\partial u/\partial y)\,\delta y\,\delta t}{\delta y} = \frac{\partial u}{\partial y}\delta t$$

$$\omega_{OB} = \lim_{\delta t\to 0}\left[\frac{(\partial u/\partial y)\,\delta t}{\delta t}\right] = \frac{\partial u}{\partial y}$$

rotation, ω_z,

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$$

$$\boldsymbol{\omega} = \tfrac{1}{2}\,\text{curl}\,\mathbf{V} = \tfrac{1}{2}\nabla\times\mathbf{V}$$

$$\frac{1}{2}\nabla\times\mathbf{V} = \frac{1}{2}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u & v & w\end{vmatrix}$$

$$= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\mathbf{i}} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\mathbf{j}} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\mathbf{k}}$$

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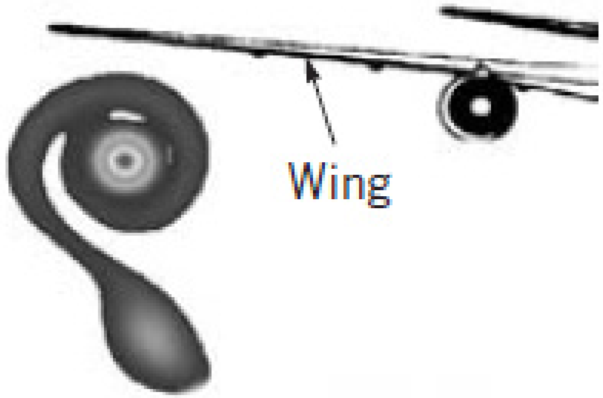
The ***vorticity***, ζ, is defined as ➡ $\zeta = 2\,\boldsymbol{\omega} = \nabla \times \mathbf{V}$

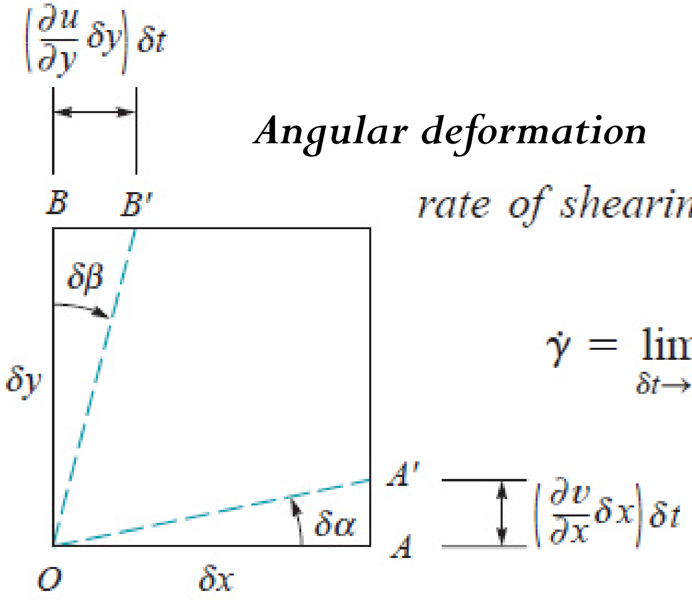


rotate about the z axis as an *undeformed* block (i.e., $\omega_{OA} = -\omega_{OB}$) only when $\partial u/\partial y = -\partial v/\partial x$

when $\partial u/\partial y = \partial v/\partial x$ ➡ $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$

More generally if $\nabla \times \mathbf{V} = 0$, ➡ then the rotation (and the vorticity) are zero

flow fields ➡ ***irrotational.***



Angular deformation

the shearing strain, $\delta\gamma$, ➡ $\delta\gamma = \delta\alpha + \delta\beta$

rate of shearing strain or the *rate of angular deformation*

$$\dot{\gamma} = \lim_{\delta t \to 0} \frac{\delta\gamma}{\delta t} = \lim_{\delta t \to 0}\left[\frac{(\partial v/\partial x)\,\delta t + (\partial u/\partial y)\,\delta t}{\delta t}\right] \quad ➡ \quad \dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$\partial u/\partial y = -\partial v/\partial x$, the rate of angular deformation is zero

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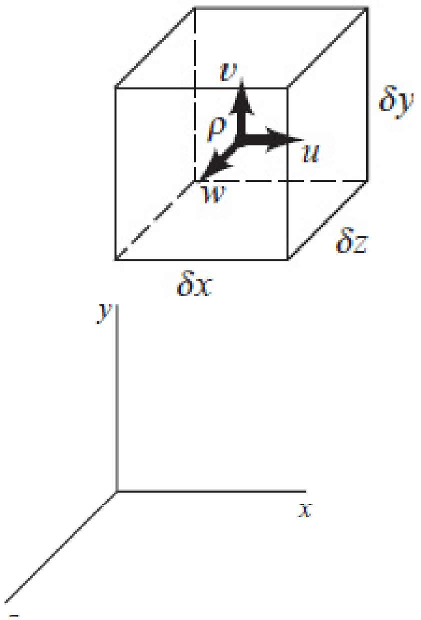
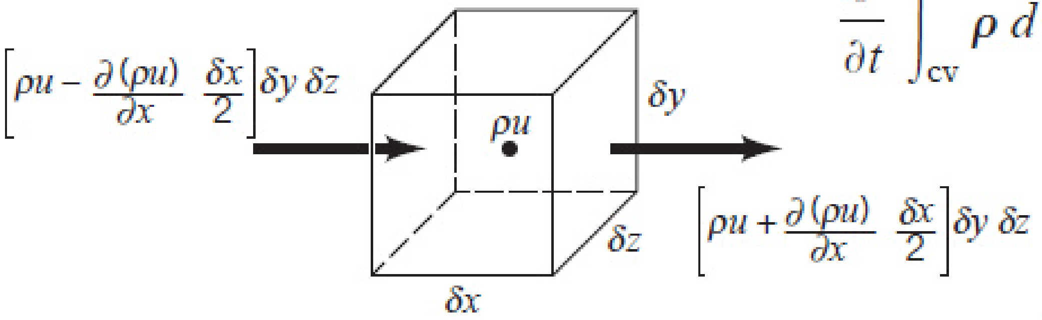
Conservation of Mass

$$\frac{DM_{\text{sys}}}{Dt} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}\int_{\text{cv}} \rho \, d\forall + \int_{\text{cs}} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA = 0$$





$$\frac{\partial}{\partial t}\int_{\text{cv}} \rho \, d\forall \approx \frac{\partial \rho}{\partial t}\,\delta x\,\delta y\,\delta z$$

$$\rho u|_{x+(\delta x/2)} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2} \qquad \rho u|_{x-(\delta x/2)} = \rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}$$

$$\begin{array}{c}\text{Net rate of mass}\\ \text{outflow in } x \text{ direction}\end{array} = \left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\,\delta z - \left[\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\,\delta z = \frac{\partial(\rho u)}{\partial x}\,\delta x\,\delta y\,\delta z$$

$$\begin{array}{c}\text{Net rate of mass}\\ \text{outflow in } y \text{ direction}\end{array} = \frac{\partial(\rho v)}{\partial y}\,\delta x\,\delta y\,\delta z \qquad \begin{array}{c}\text{Net rate of mass}\\ \text{outflow in } z \text{ direction}\end{array} = \frac{\partial(\rho w)}{\partial z}\,\delta x\,\delta y\,\delta z$$

$$\begin{array}{c}\text{Net rate of}\\ \text{mass outflow}\end{array} = \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\delta x\,\delta y\,\delta z \quad \Rightarrow \quad \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0}$$

valid for steady or unsteady flow
compressible or incompressible

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$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \equiv \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

For *steady* → $\nabla \cdot \rho \mathbf{V} = 0$

For *incompressible* fluids → ρ, is a constant → $\nabla \cdot \mathbf{V} = 0$ → $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

مثال: سرعت w را طوری بیابید که سیال تراکم ناپذیر باشد.

$$u = x^2 + y^2 + z^2$$
$$v = xy + yz + z$$
$$w = ?$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$\frac{\partial u}{\partial x} = 2x$ and $\frac{\partial v}{\partial y} = x + z$ → $\frac{\partial w}{\partial z} = -2x - (x + z) = -3x - z$ → $w = -3xz - \frac{z^2}{2} + f(x, y)$

Cylindrical Polar Coordinates

$$\mathbf{V} = v_r \hat{\mathbf{e}}_r + v_\theta \hat{\mathbf{e}}_\theta + v_z \hat{\mathbf{e}}_z$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

For steady, compressible flow

$$\frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

For incompressible fluids (for steady or unsteady flow)

$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

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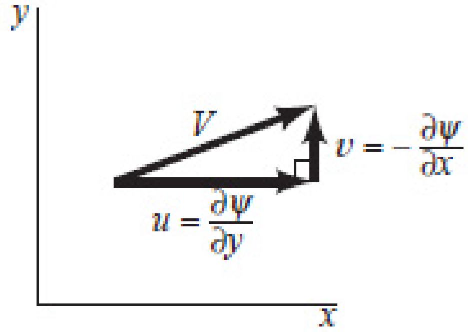
The Stream Function

$\psi(x, y)$, called the ***stream function***,

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial^2 \psi}{\partial y\, \partial x} = 0$$

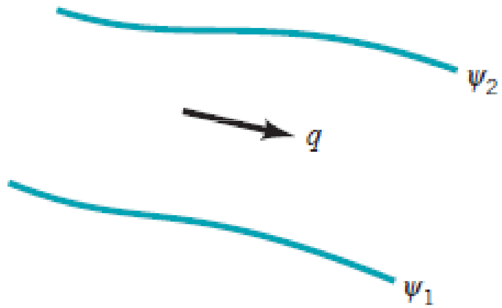


تابع جریان برای هر خط جریان یک مقدار ثابت است.

$$\frac{dy}{dx} = \frac{v}{u}$$

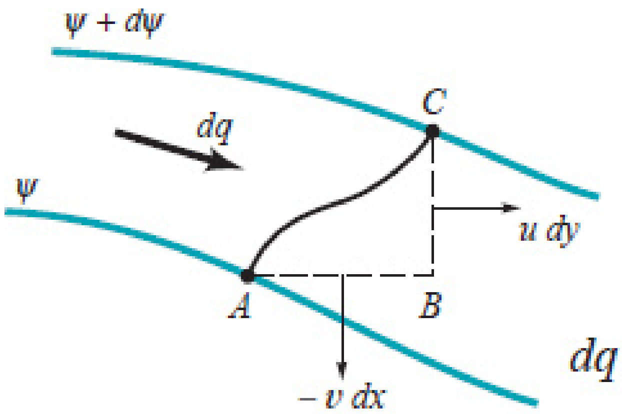
The change in the value of ψ from one point (x, y) to a nearby point $(x + dx, y + dy)$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v\, dx + u\, dy$$



Along a line of constant ψ we have $d\psi = 0$ ➡ $-v\, dx + u\, dy = 0$

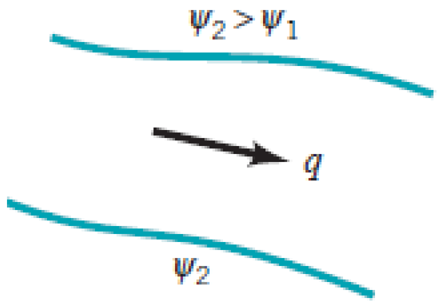
therefore, along a line of constant ψ ➡ $\dfrac{dy}{dx} = \dfrac{v}{u}$

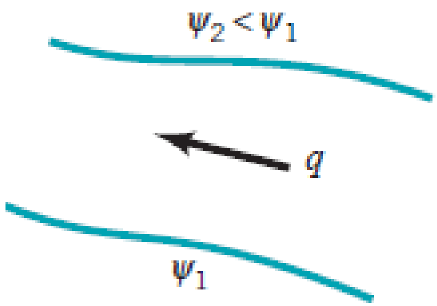


$$dq = \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx \;\Rightarrow\; dq = d\psi \;\Rightarrow\; q = \int_{\psi_1} d\psi = \psi_2 - \psi_1$$

مقدار دبی عبوری از
بین دو خط جریان

$$dq = u\, dy - v\, dx$$





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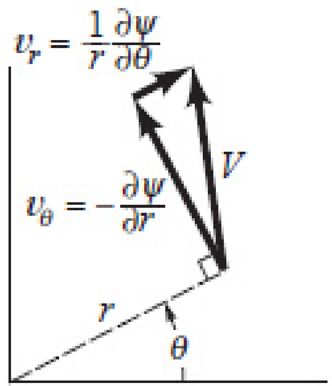
In cylindrical coordinates $\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$



the stream function, $\psi(r, \theta)$

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad v_\theta = -\frac{\partial\psi}{\partial r}$$

مثال: برای میدان سرعت داده شده برای یک سیال دایم و تراکم ناپذیر، تابع جریان را بدست آورده و خطوط جریان به همراه مسیر حرکت سیال را مشخص نمایید.

$u = 2y$

$v = 4x$

$u = \frac{\partial\psi}{\partial y} = 2y \rightarrow \psi = y^2 + f_1(x)$

$v = -\frac{\partial\psi}{\partial x} = 4x \rightarrow \psi = -2x^2 + f_2(y)$

$\psi = -2x^2 + y^2 + C$

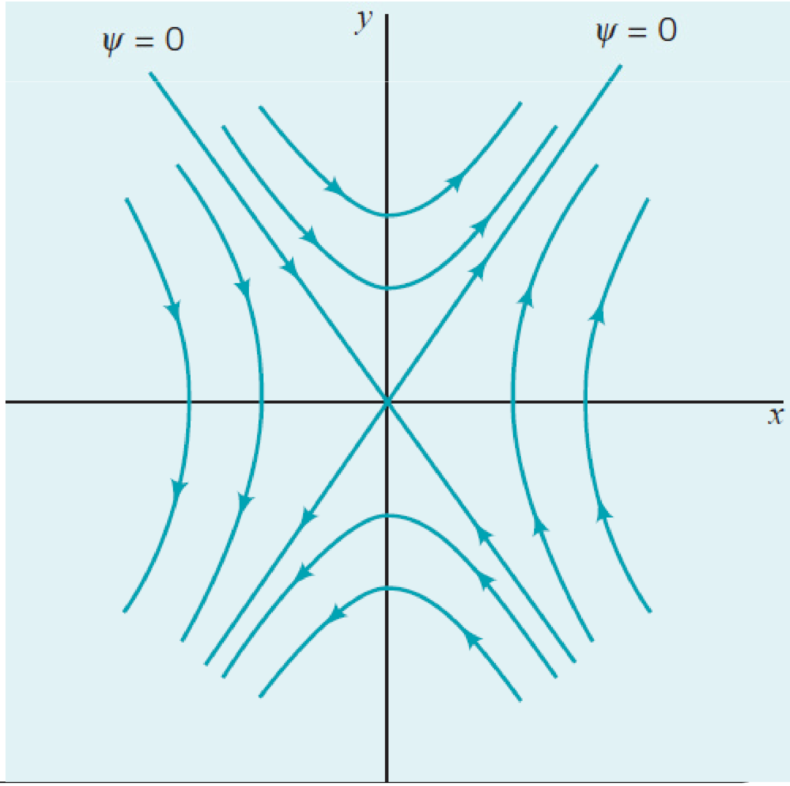
for simplicity

$C = 0 \rightarrow \psi = -2x^2 + y^2$

حال می توان به ازای مقادیر مختلف تابع جریان، خطوط جریان را ترسیم نمود

$\psi = 0 \qquad 0 = -2x^2 + y^2$

$y = \pm\sqrt{2}x$



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Conservation of Linear Momentum

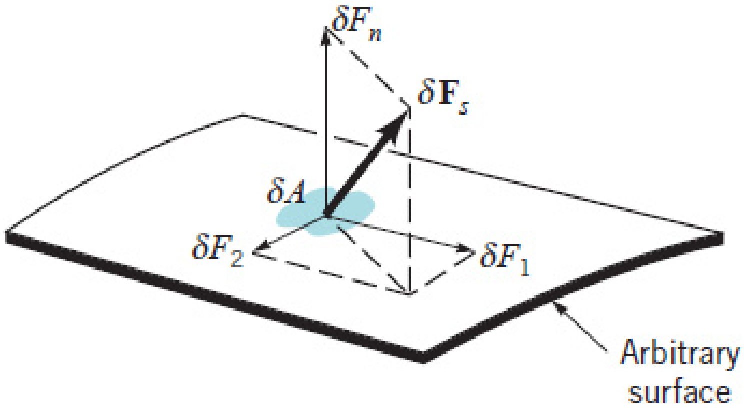
$$\delta \mathbf{F} = \frac{D(\mathbf{V}\,\delta m)}{Dt} \;\rightarrow\; \delta \mathbf{F} = \delta m \frac{D\mathbf{V}}{Dt} \;\rightarrow\; \delta \mathbf{F} = \delta m\, \mathbf{a}$$

$D\mathbf{V}/Dt$ is the acceleration, $\mathbf{a}$

surface forces

$$\delta \mathbf{F} = \delta \mathbf{F}_s + \delta \mathbf{F}_b$$

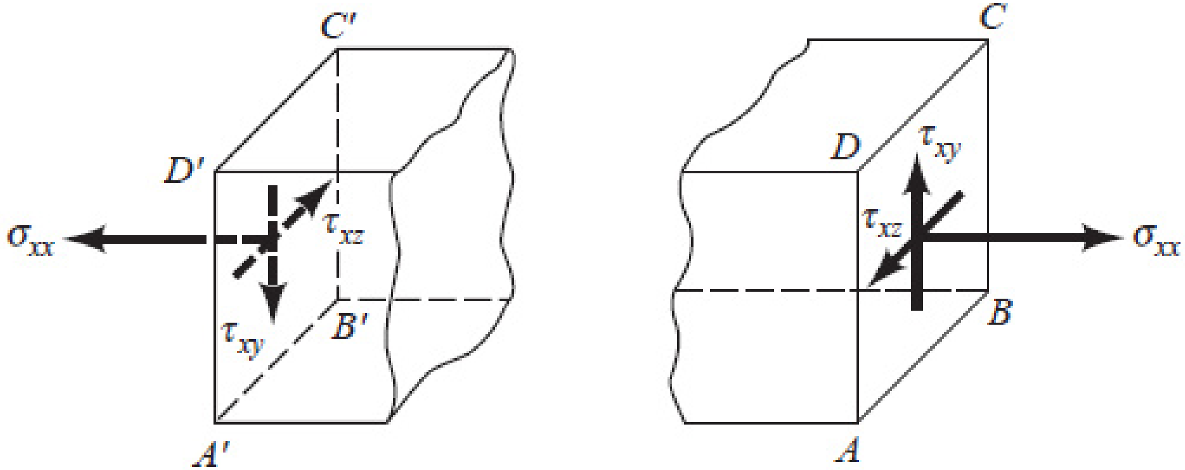
body forces



$$\delta \mathbf{F}_b = \delta m\, \mathbf{g} \begin{cases} \delta F_{bx} = \delta m\, g_x \\ \delta F_{by} = \delta m\, g_y \\ \delta F_{bz} = \delta m\, g_z \end{cases}$$

normal stress, σ_n $\qquad \sigma_n = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A}$

shearing stresses $\begin{cases} \tau_1 = \lim_{\delta A \to 0} \frac{\delta F_1}{\delta A} \\ \tau_2 = \lim_{\delta A \to 0} \frac{\delta F_2}{\delta A} \end{cases}$



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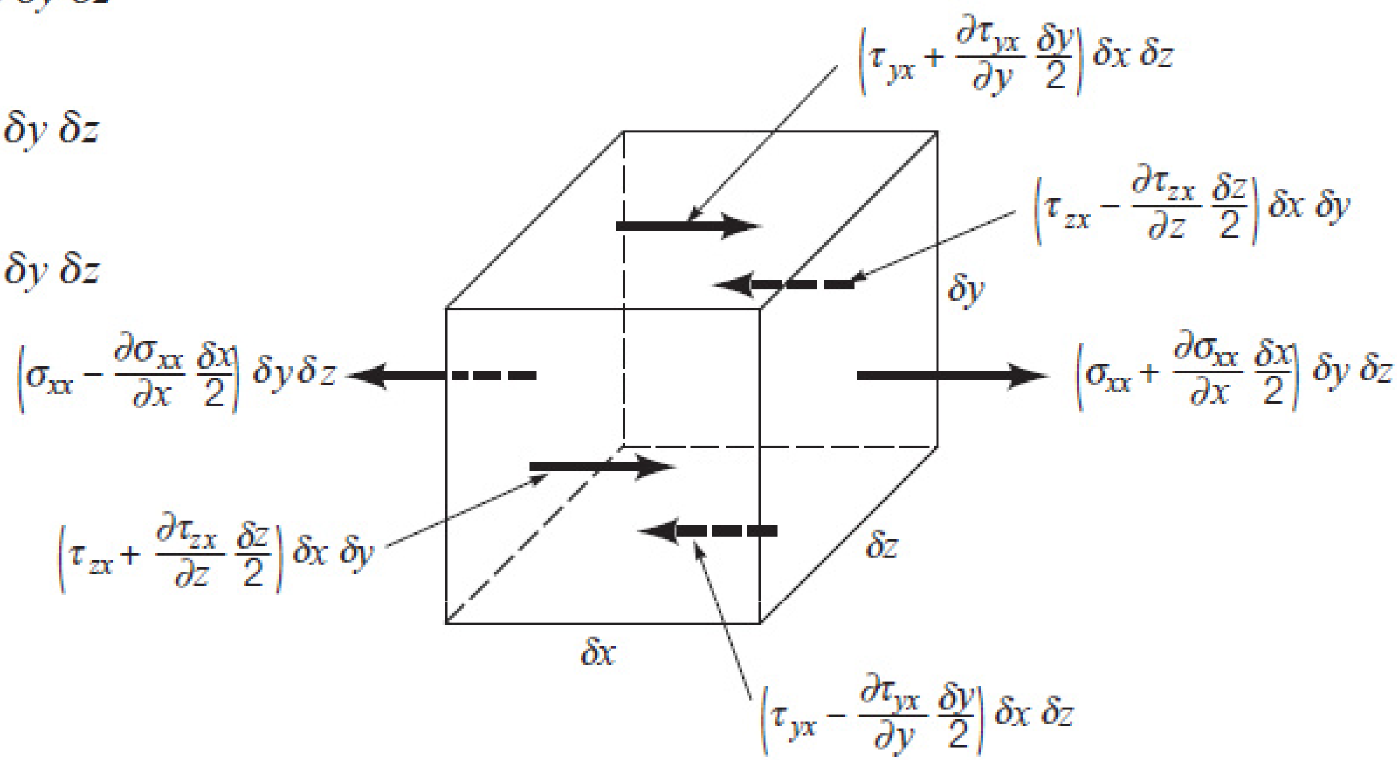
$$\delta F_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x\, \delta y\, \delta z$$

$$\delta F_{sy} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)\delta x\, \delta y\, \delta z$$

$$\delta F_{sz} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)\delta x\, \delta y\, \delta z$$

$$\delta \mathbf{F}_s = \delta F_{sx}\hat{\mathbf{i}} + \delta F_{sy}\hat{\mathbf{j}} + \delta F_{sz}\hat{\mathbf{k}}$$

$$\delta \mathbf{F} = \delta \mathbf{F}_s + \delta \mathbf{F}_b$$



$$\delta F_x = \delta m\, a_x$$

$$\delta F_y = \delta m\, a_y$$

$$\delta F_z = \delta m\, a_z$$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Inviscid Flow

در جریان غیر لزج اثر تنش برشی قابل صرفنظر کردن است لذا تنها تنش موثر، تنش نرمال می باشد

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

$$-p = \sigma_{xx} = \sigma_{yy} = \sigma_{zz}$$

Euler's equations of motion

$$\rho \mathbf{g} - \nabla p = \rho\left[\frac{\partial \mathbf{V}}{\partial t} + \underbrace{(\mathbf{V}\cdot\nabla)\mathbf{V}}_{\text{nonlinear}}\right]$$

$$\begin{cases} \rho g_x - \dfrac{\partial p}{\partial x} = \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z}\right) \\ \rho g_y - \dfrac{\partial p}{\partial y} = \rho\left(\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z}\right) \\ \rho g_z - \dfrac{\partial p}{\partial z} = \rho\left(\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z}\right) \end{cases}$$

Bernoulli equation

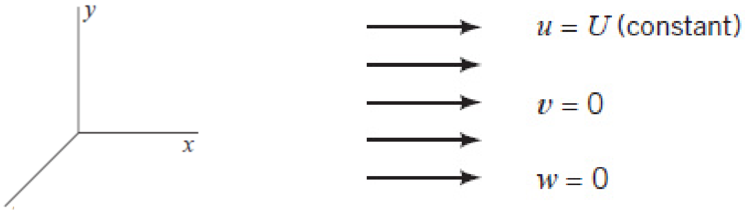
$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

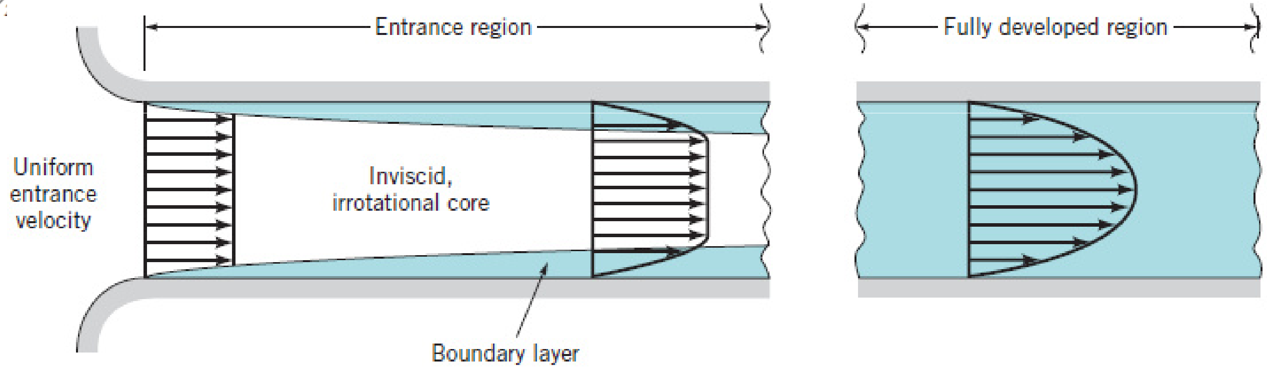
- inviscid flow
- steady flow
- incompressible flow
- flow along a streamline

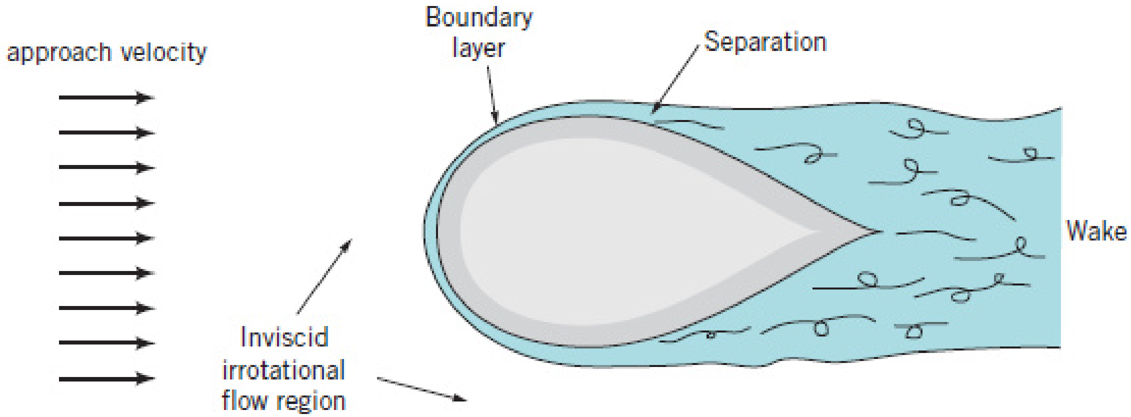
Irrotational Flow

$$\nabla \times \mathbf{V} = 0 \rightarrow \tfrac{1}{2}(\nabla \times \mathbf{V}) = 0 \rightarrow \begin{cases} \omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0 \\ \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0 \\ \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \end{cases}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \rightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \rightarrow \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$$







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The Bernoulli Equation for Irrotational Flow

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

- inviscid flow
- incompressible flow
- steady flow
- irrotational flow

نکته: در شرایطی که میدان سیال غیر چرخشی باشد، رابطه برنولی را می توان برای هر دو نقطه از میدان سیال نوشت و ضرورتی بر وجود دو نقطه بر یک خط جریان نیست.

For an irrotational flow the velocity components can be expressed scalar function $\phi(x, y, z, t)$

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z}$$

ϕ is called the ***velocity potential.***

$$\mathbf{V} = \nabla \phi$$

For an incompressible fluid

$$\nabla \cdot \mathbf{V} = 0$$

$$\nabla^2 \phi = 0 \Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Laplace's equation

$$\nabla^2(\ ) = \nabla \cdot \nabla(\ )$$

Laplacian operator

inviscid, incompressible, irrotational flow fields ***potential flow***

cylindrical coordinates, r, θ, and z.

$$\nabla \phi = \frac{\partial \phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{\partial \phi}{\partial z}\hat{\mathbf{e}}_z$$

$$v_r = \frac{\partial \phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \qquad v_z = \frac{\partial \phi}{\partial z}$$

$$\nabla(\ ) = \frac{\partial(\ )}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial(\ )}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{\partial(\ )}{\partial z}\hat{\mathbf{e}}_z$$

$$\mathbf{V} = v_r\hat{\mathbf{e}}_r + v_\theta\hat{\mathbf{e}}_\theta + v_z\hat{\mathbf{e}}_z$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

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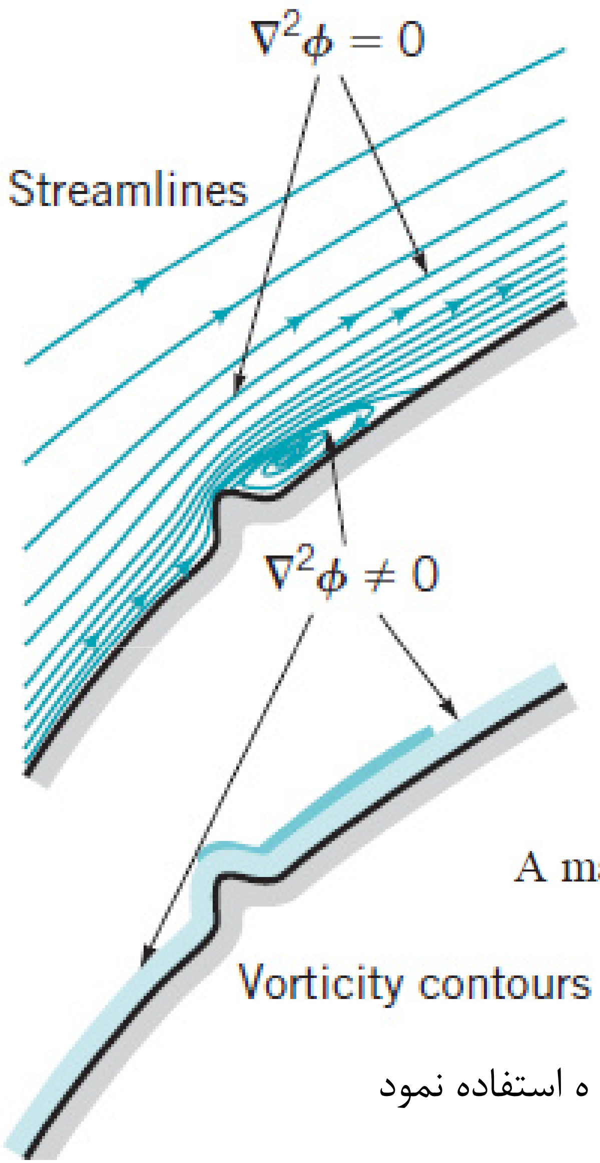
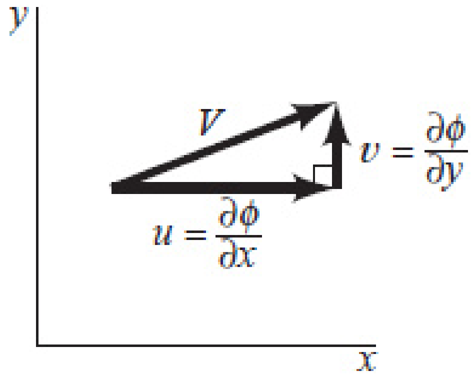
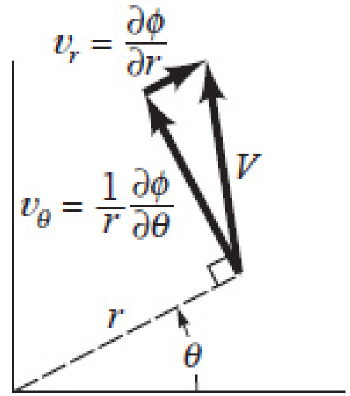




$$u = \frac{\partial \phi}{\partial x}$$

$$v = \frac{\partial \phi}{\partial y}$$



$$v_r = \frac{\partial \phi}{\partial r}$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$u = \frac{\partial \psi}{\partial y} \qquad v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$v = -\frac{\partial \psi}{\partial x} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$

irrotationality $\Rightarrow$ $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

$$\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) \Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

A major advantage of Laplace's equation is a linear partial differential equation

if $\phi_1(x, y, z)$ and $\phi_2(x, y, z)$ are two solutions to Laplace's equation,

$\phi_3 = \phi_1 + \phi_2$ is also a solution

این نکته بدین معنی است که می توان برای حل مسایل جریان لاپلاس از ترکیب چند مسئله ساد ه استفاده نمود

irrotational flow

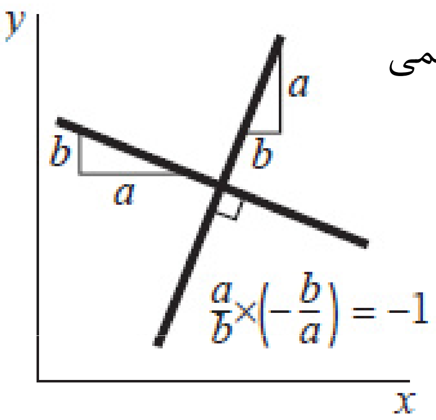
$$\left.\frac{dy}{dx}\right|_{\text{along } \psi = \text{constant}} = \frac{v}{u}$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = u\,dx + v\,dy$$

Along a line of constant ϕ we have $d\phi = 0$

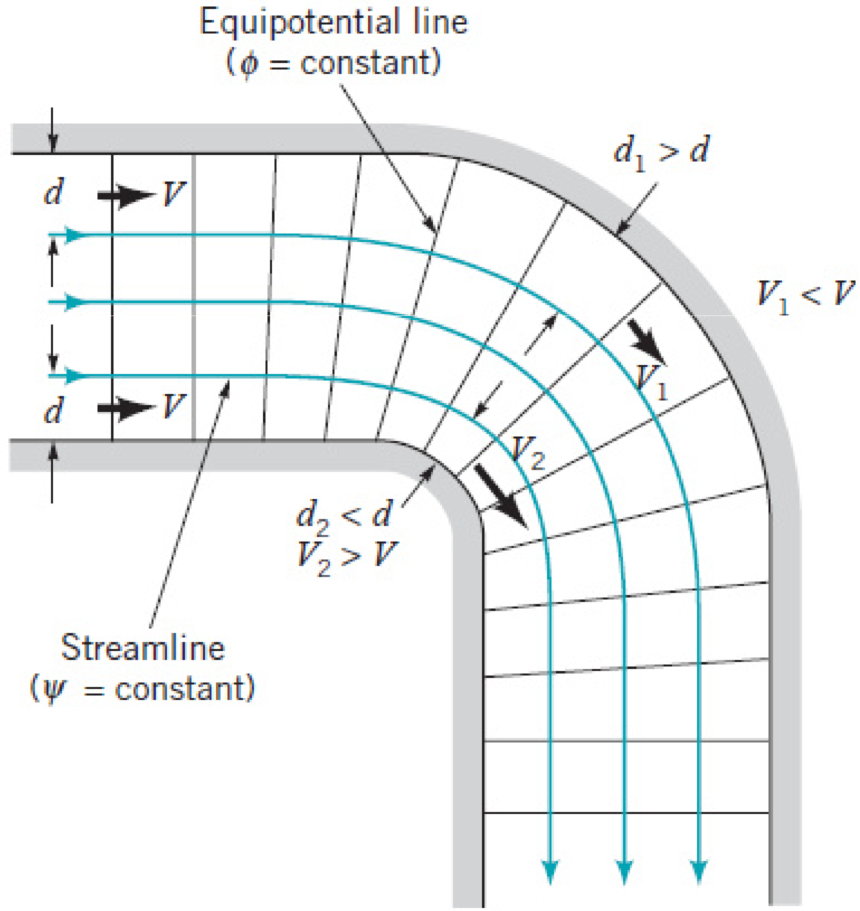
$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{\text{along } \phi = \text{constant}} = -\frac{u}{v} \qquad \left.\frac{dy}{dx}\right|_{\text{along } \psi = \text{constant}} = \frac{v}{u}$$

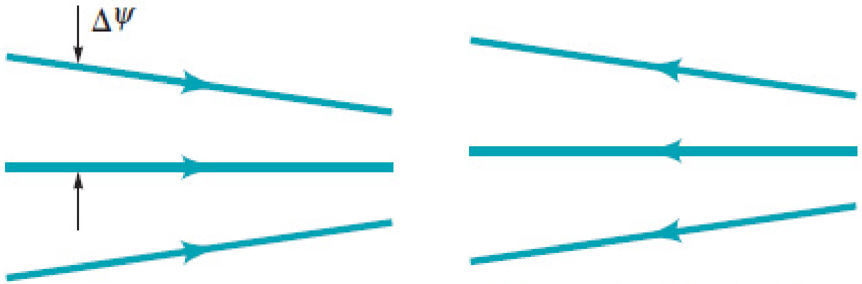
دو خط بر هم عمود بوده که در این صورت
حاصلضرب شیب دو خط برابر ۱- خواهد بود



$$\frac{a}{b}\times\left(-\frac{b}{a}\right) = -1$$

در این صورت می توان با استفاده از روش ترسیمی
تابع جریان و پتانسیل را ترسیم نمود





Streamwise acceleration

Streamwise deceleration

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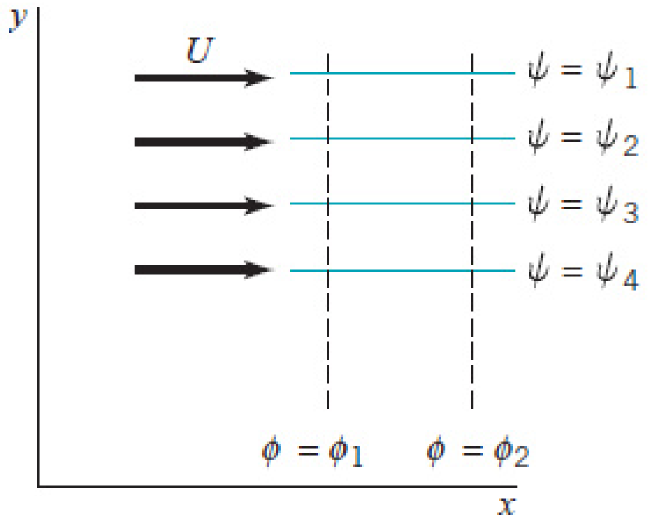
Uniform Flow

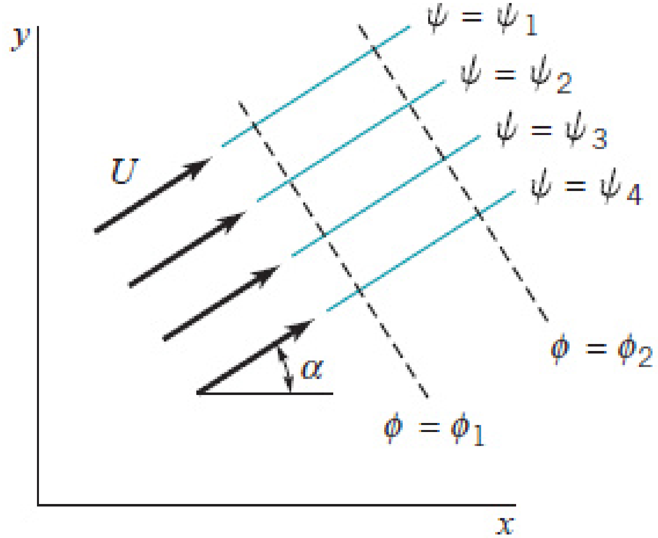
the streamlines are all straight and parallel magnitude of the velocity is constant



$u = U$ and $v = 0$ → $\frac{\partial \phi}{\partial x} = U$ $\frac{\partial \phi}{\partial y} = 0$ → $\phi = Ux + C$

ثابت دلخواه = • ← $\phi = Ux$

$$\frac{\partial \psi}{\partial y} = U \qquad \frac{\partial \psi}{\partial x} = 0 \rightarrow \psi = Uy$$



$$\phi = U(x \cos \alpha + y \sin \alpha)$$

$$\psi = U(y \cos \alpha - x \sin \alpha)$$

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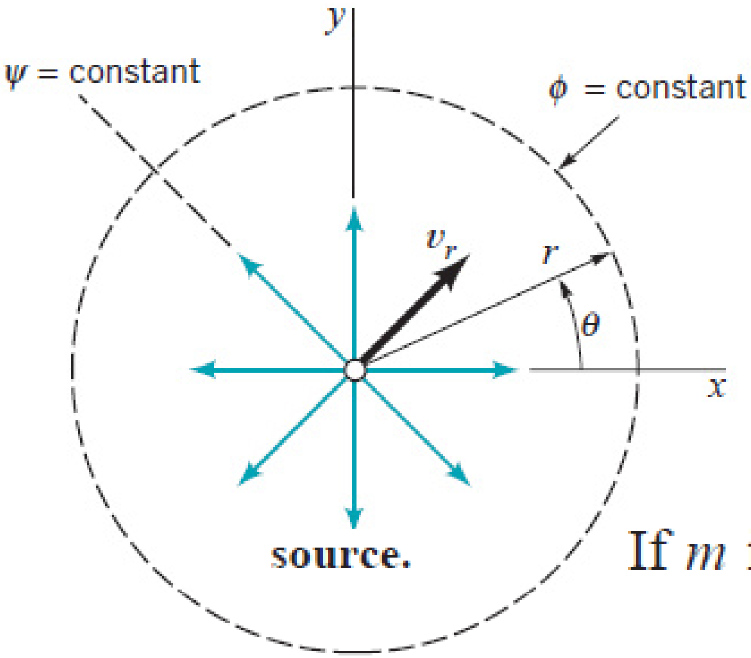
Source and Sink چشمه و چاه

m be the volume rate of flow per unitlength نرخ دبی حجمی بر واحد طول



$$(2\pi r)v_r = m \Rightarrow v_r = \frac{m}{2\pi r}$$

$$v_\theta = 0$$

$$\frac{\partial\phi}{\partial r} = \frac{m}{2\pi r} \qquad \frac{1}{r}\frac{\partial\phi}{\partial\theta} = 0 \Rightarrow \phi = \frac{m}{2\pi}\ln r$$

If m is
- positive → flow is considered to be a *source* flow
- negative → the flow is considered to be a *sink* flow.

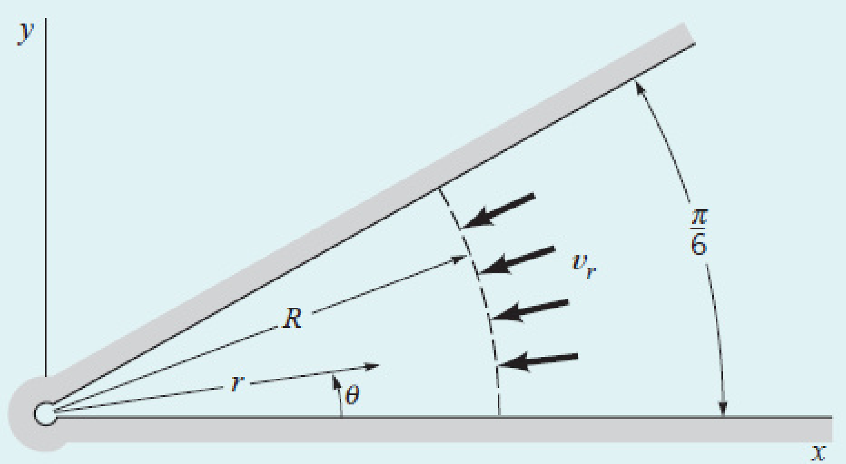
نکته: در r=0 مقدار سرعت تعریف نشده می باشد که این مقدار با واقعیت فیزیکی تطبیق ندارد. این نشان دهنده این مطلب است که جریان چشمه و چاه به صورت واقعی وجود نداشته و از نظر ریاضی یک نقطه تکینی می باشد.

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{m}{2\pi r}$$

$$v_\theta = -\frac{\partial\psi}{\partial r} = 0$$

$$\Rightarrow \psi = \frac{m}{2\pi}\theta$$

مثال: برای یک جریان غیر لزج، تراکم ناپذیر و دائم در یک گوه که دارای یک سوراخ کوچک در انتها می باشد، تابع پتانسیل سرعت به صورت زیر داده شده است. مقدار دبی گذرنده از خروجی را بدست آورید.



$$\phi = -2 \ln r$$

$$v_r = \frac{\partial \phi}{\partial r} = -\frac{2}{r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

we have a purely radial flow

$$q = \int_0^{\pi/6} v_r R\, d\theta = -\int_0^{\pi/6} \left(\frac{2}{R}\right) R\, d\theta = -\frac{\pi}{3} = -1.05 \text{ ft}^2/\text{s}$$

Vortex

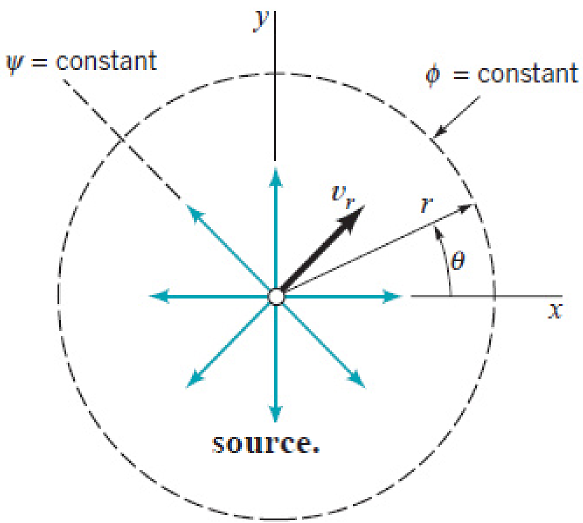
شبیه رفتار چشمه با جابجایی تابع پتانسیل و جریان با یکدیگر

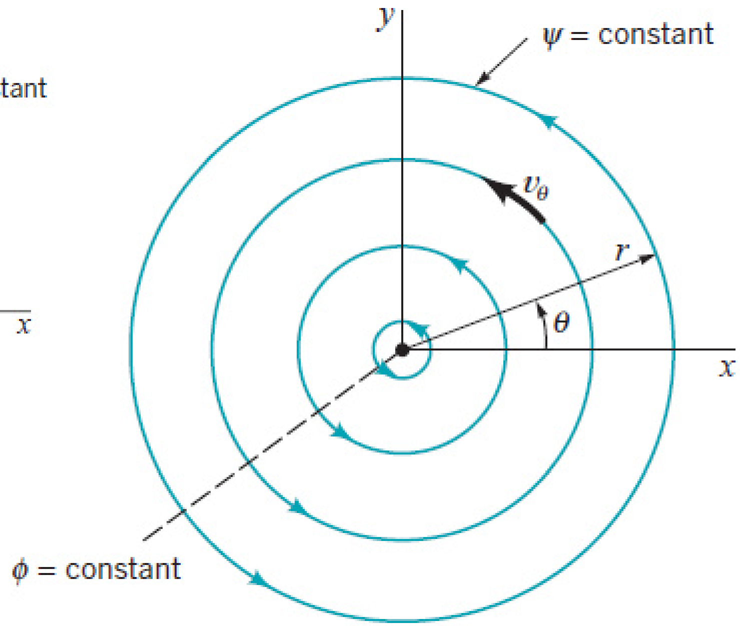
$$\phi = K\theta$$

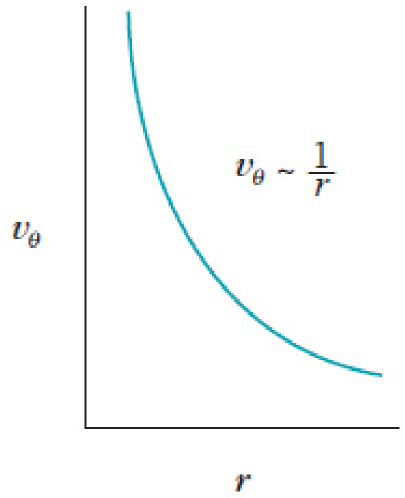
$$\psi = -K \ln r$$

$$v_r = 0$$

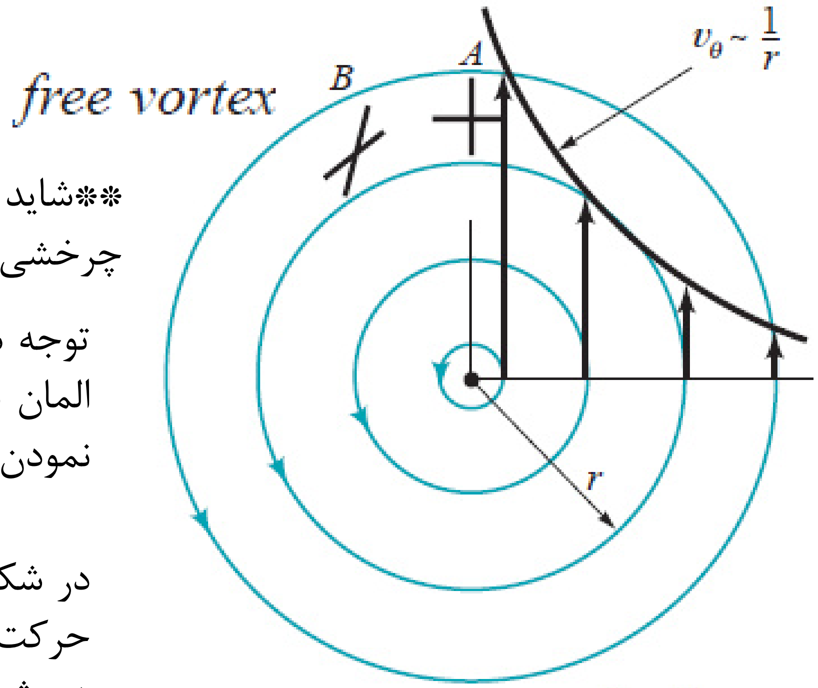
$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{K}{r}$$







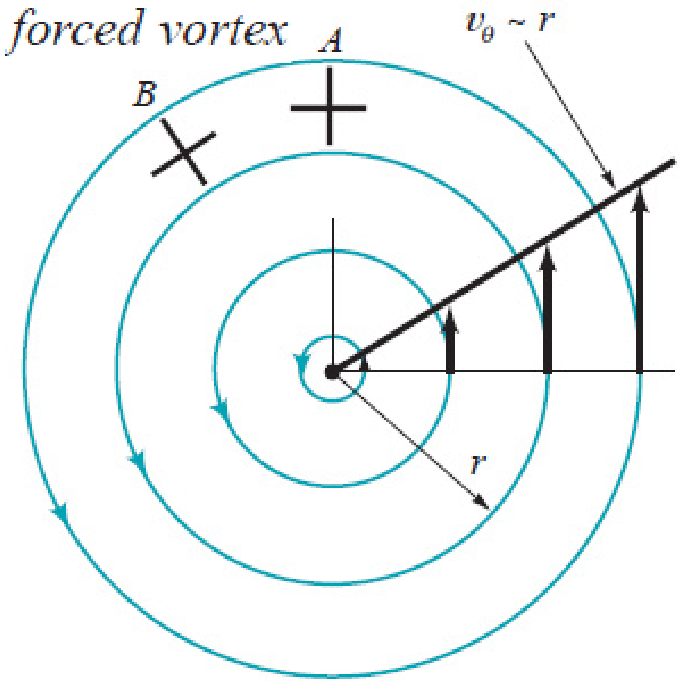
at $r = 0$ ➡ $v_\theta = \text{infinite}$



**شاید به نظر عجیب باشد که میدان گردابه غیر چرخشی باشد...

توجه شود که جریان چرخشی باعث می شود تا المان شروع به چرخش نماید و به معنای دنبال نمودن جریان توسط المان نمی باشد

در شکل اول، علامتی که در یک شعاع ثابت قرار دارد (افقی) مماس بر خطوط جریان حرکت نموده و علامت عمودی به واسطه سرعت زاویه ای مختلف(بدلیل قرار گرفتن در شعاعهای متفاوت) دارای سرعت مختلف بوده و به همین علت تغییرات فوق مشاهده می گردد. اگرچه هر دو علامت می چرخند اما متوسط سرعت زاویه ای دو علامت صفر می باشد، لذا جریان غیر چرخشی است.



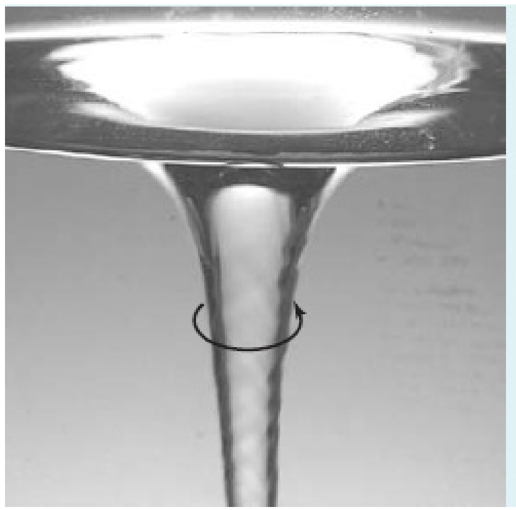
A *combined vortex*

a forced vortex as a central core

a free vortex outside the core

$$v_\theta = \omega r \qquad r \le r_0$$

$$v_\theta = \frac{K}{r} \qquad r > r_0$$



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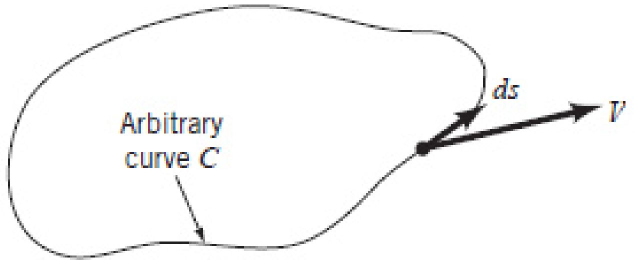


circulation

$$\Gamma = \oint_C \mathbf{V} \cdot d\mathbf{s}$$

$$\mathbf{V} = \nabla\phi \text{ so that } \mathbf{V} \cdot d\mathbf{s} = \nabla\phi \cdot d\mathbf{s} = d\phi$$

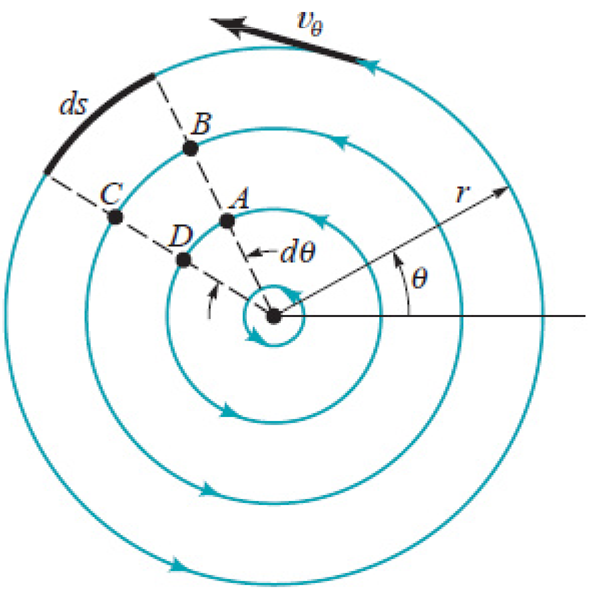
$$\Gamma = \oint_C d\phi = 0$$

for an irrotational flow the circulation will generally be zero

در صورتی که مسئله دارای نقطه تکین باشد چرخش برابر صفر نخواهد شد.

for the free vortex with $v_\theta = K/r$ $\quad \Gamma = \int_0^{2\pi} \frac{K}{r}(r\, d\theta) = 2\pi K$

which shows that the circulation is nonzero and the constant $K = \Gamma/2\pi$



در صورتی که ناحیه مورد نظر دارای نقطه تکین نباشد مانند ناحیه ABCD ، چرخش برابر صفر است.

for the free vortex

$$\phi = \frac{\Gamma}{2\pi}\theta \qquad \psi = -\frac{\Gamma}{2\pi}\ln r$$

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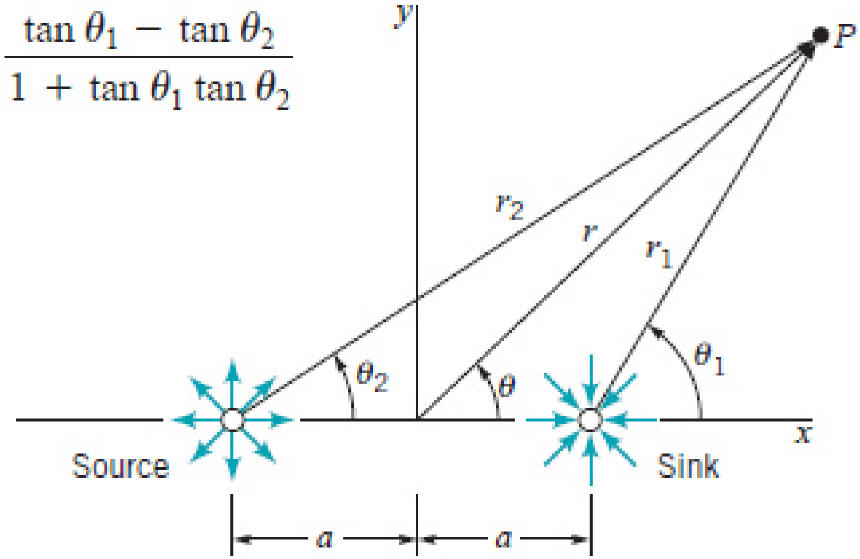
Doublet

$$\psi = -\frac{m}{2\pi}(\theta_1 - \theta_2)$$

$$\tan\left(-\frac{2\pi\psi}{m}\right) = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

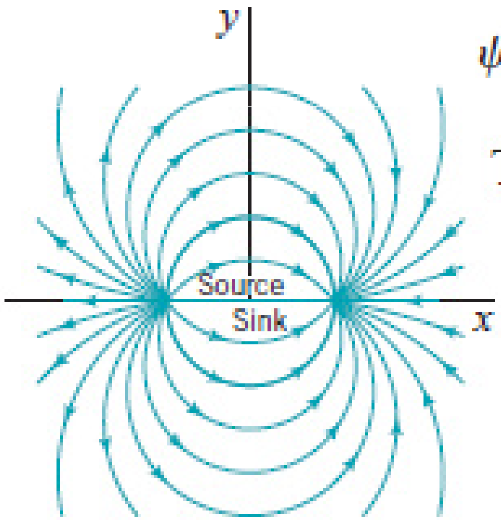
$$\tan\theta_1 = \frac{r\sin\theta}{r\cos\theta - a}$$

$$\tan\theta_2 = \frac{r\sin\theta}{r\cos\theta + a}$$



$$\tan\left(-\frac{2\pi\psi}{m}\right) = \frac{2ar\sin\theta}{r^2 - a^2}$$

$$\psi = -\frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$



$$\psi = -\frac{m}{2\pi}\frac{2ar\sin\theta}{r^2 - a^2} = -\frac{mar\sin\theta}{\pi(r^2 - a^2)}$$

The so-called ***doublet*** is formed by letting the source and sink approach

$(a \rightarrow 0)$ ma/π remains constant

$(m \rightarrow \infty)$ $r/(r^2 - a^2) \rightarrow 1/r$

$$\psi = -\frac{K\sin\theta}{r}$$

$$\phi = \frac{K\cos\theta}{r}$$

where K, a constant equal to ma/π, is called the *strength* of the doublet.

Summary of Basic, Plane Potential Flows

Description of Flow Field	Velocity Potential	Stream Function	Velocity Components[a]
Uniform flow at angle α with the x axis (see Fig. 6.16*b*)	$\phi = U(x \cos \alpha + y \sin \alpha)$	$\psi = U(y \cos \alpha - x \sin \alpha)$	$u = U \cos \alpha$ $v = U \sin \alpha$
Source or sink (see Fig. 6.17) $m > 0$ source $m < 0$ sink	$\phi = \dfrac{m}{2\pi} \ln r$	$\psi = \dfrac{m}{2\pi} \theta$	$v_r = \dfrac{m}{2\pi r}$ $v_\theta = 0$
Free vortex (see Fig. 6.18) $\Gamma > 0$ counterclockwise motion $\Gamma < 0$ clockwise motion	$\phi = \dfrac{\Gamma}{2\pi} \theta$	$\psi = -\dfrac{\Gamma}{2\pi} \ln r$	$v_r = 0$ $v_\theta = \dfrac{\Gamma}{2\pi r}$
Doublet (see Fig. 6.23)	$\phi = \dfrac{K \cos \theta}{r}$	$\psi = -\dfrac{K \sin \theta}{r}$	$v_r = -\dfrac{K \cos \theta}{r^2}$ $v_\theta = -\dfrac{K \sin \theta}{r^2}$

[a]Velocity components are related to the velocity potential and stream function through the relationships:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}.$$

Viscous Flow

In cylindrical polar coordinates

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\sigma_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r}$$

$$\sigma_{\theta\theta} = -p + 2\mu \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)$$

$$\tau_{zr} = \tau_{rz} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$

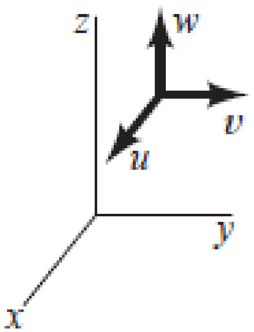
روابط بالا را در معادله جایگذاری نماییم

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

The Navier–Stokes Equations



(x direction) $$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

(y direction) $$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

(z direction) $$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

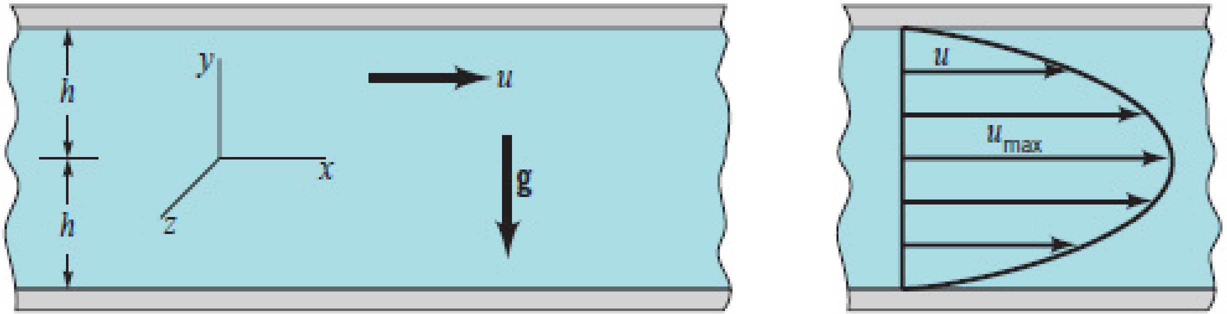
(r direction) $$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)$$
$$=-\frac{\partial p}{\partial r}+\rho g_r+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]$$

(θ direction) $$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)$$
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\rho g_\theta+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)-\frac{v_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]$$

(z direction) $$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$
$$=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]$$

Steady, Laminar Flow between Fixed Parallel Plates

$v = 0$ and $w = 0$ steady flow $\partial u/\partial t = 0$



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \;\rightarrow\; \partial u/\partial x = 0.$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \rightarrow 0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \rightarrow 0 = -\frac{\partial p}{\partial y} - \rho g \qquad g_y = -g$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \rightarrow 0 = -\frac{\partial p}{\partial z}$$

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} \rightarrow \frac{du}{dy} = \frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right)y + c_1 \rightarrow u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 + c_1 y + c_2$$

$u = 0$ for $y = \pm h$ → $c_1 = 0$

no-slip condition

$$c_2 = -\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)h^2 \rightarrow u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - h^2)$$

$$q = \int_{-h}^{h} u\,dy = \int_{-h}^{h} \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - h^2)\,dy = -\frac{2h^3}{3\mu}\left(\frac{\partial p}{\partial x}\right)$$

$$\frac{\Delta p}{\ell} = -\frac{\partial p}{\partial x} \rightarrow q = \frac{2h^3 \Delta p}{3\mu\ell}$$

mean velocity, V:

$$V = \frac{h^2 \Delta p}{3\mu\ell}$$

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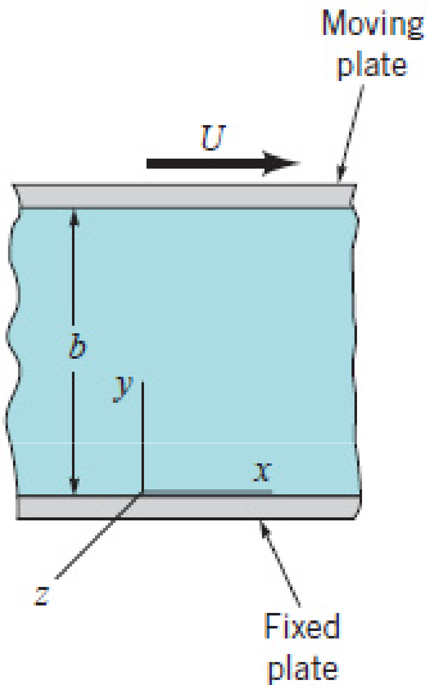
$$u_{max} = -\frac{h^2}{2\mu}\left(\frac{\partial p}{\partial x}\right) \qquad u_{max} = \tfrac{3}{2}V$$

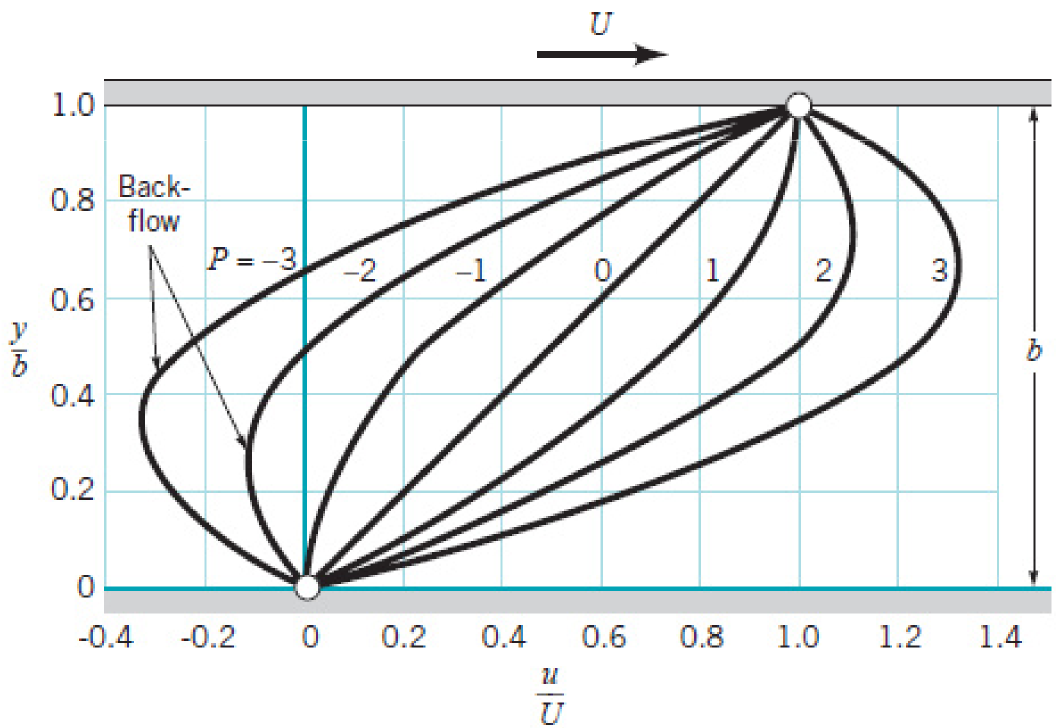
Couette Flow



$u = 0$ at $y = 0$ and $u = U$ at $y = b$

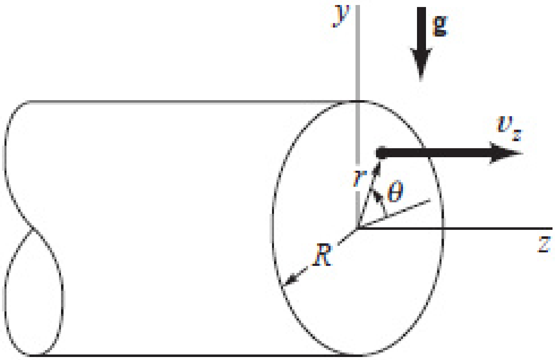
$$u = U\frac{y}{b} + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - by)$$

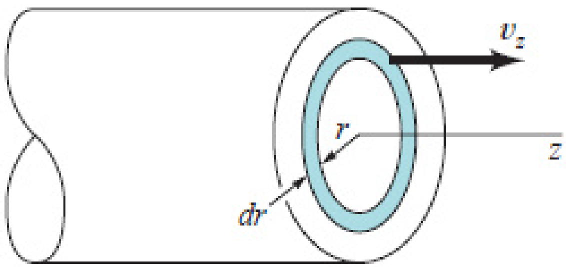
$$\frac{u}{U} = \frac{y}{b} - \frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)\left(\frac{y}{b}\right)\left(1 - \frac{y}{b}\right)$$

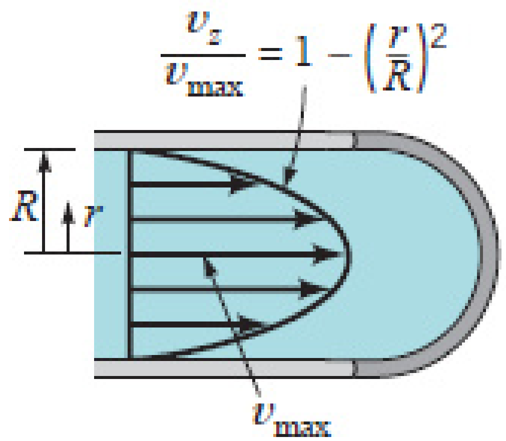


Steady, Laminar Flow in Circular Tubes

$$v_z = v_z(r)$$







$$0 = -\rho g \sin\theta - \frac{\partial p}{\partial r} \qquad p = -\rho g y + f_1(z)$$

$$0 = -\rho g \cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{\mu}\frac{\partial p}{\partial z}$$

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)r^2 + c_1 \ln r + c_2$$

$$c_1 = 0 \qquad c_2 = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)R^2$$

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r^2 - R^2)$$

$$dQ = v_z(2\pi r)\,dr \Rightarrow Q = 2\pi\int_0^R v_z r\,dr$$

$$\left.\begin{aligned} Q &= -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial z}\right) \\ \frac{\Delta p}{\ell} &= -\frac{\partial p}{\partial z} \end{aligned}\right\} \quad Q = \frac{\pi R^4 \Delta p}{8\mu\ell}$$

mean velocity, V, where $V = Q/\pi R^2 \Rightarrow V = \frac{R^2 \Delta p}{8\mu\ell}$

$$v_{max} = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial z}\right) = \frac{R^2\Delta p}{4\mu\ell}$$ center of the tube $\quad v_{max} = 2V$

$$\frac{v_z}{v_{max}} = 1 - \left(\frac{r}{R}\right)^2$$

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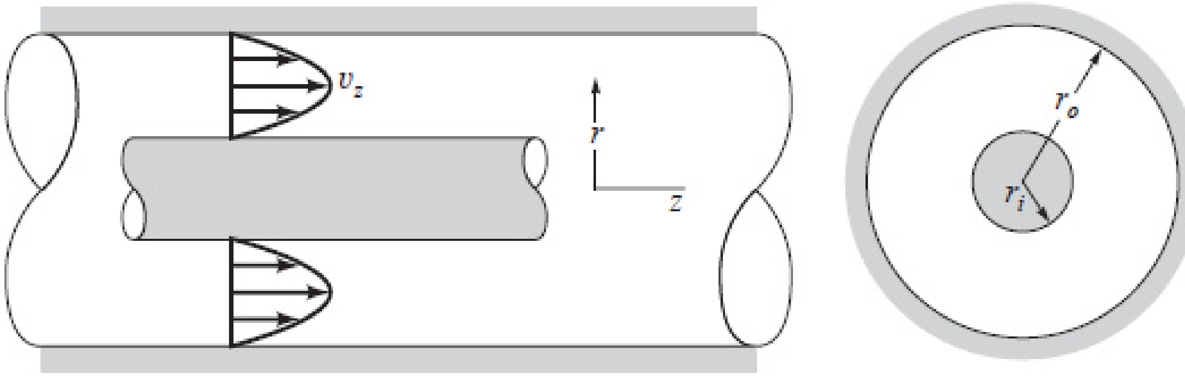
Steady, Axial, Laminar Flow in an Annulu



$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[r^2 - r_o^2 + \frac{r_i^2 - r_o^2}{\ln(r_o/r_i)}\ln\frac{r}{r_o}\right]$$

$$Q = \int_{r_i}^{r_o} v_z(2\pi r)\,dr = -\frac{\pi}{8\mu}\left(\frac{\partial p}{\partial z}\right)\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right] \Rightarrow Q = \frac{\pi \Delta p}{8\mu\ell}\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right]$$

The maximum velocity occurs at the radius $r = r_m$

$$r_m = \left[\frac{r_o^2 - r_i^2}{2\ln(r_o/r_i)}\right]^{1/2}$$

hydraulic diameter, D_h:

$$D_h = \frac{4 \times \text{cross-sectional area}}{\text{wetted perimeter}}$$

$$D_h = \frac{4\pi(r_o^2 - r_i^2)}{2\pi(r_o + r_i)} = 2(r_o - r_i)$$