ASM

• Algebraic Reynolds Stress Model is a simplified Reynolds Stress Model

$$\text{RSM}: C_{ij} - D_{ij} = P_{ij} + \Phi_{ij} - \varepsilon_{ij}$$

$$k - \varepsilon: \; C^k - D^k = P^k - \varepsilon$$

• The assumption in ASM is that the transport (convective and diffusive) of $\overline{u_i u_j}$ is related to that of k, i.e.

$$\left. \begin{array}{l} C_{ij} - D_{ij} = \dfrac{\overline{u_i u_j}}{k}\left(C^k - D^k\right) \\ k - \varepsilon: \; C^k - D^k = P^k - \varepsilon \end{array} \right\} \quad P_{ij} + \Phi_{ij} - \varepsilon_{ij} = \frac{\overline{u_i u_j}}{k}\left(P^k - \varepsilon\right)$$

Thus the transport equation (PDE) for $\overline{u_i u_j}$ has been transformed into an *algebraic* equation

$$P_{ij} + \Phi_{ij} - \varepsilon_{ij} = \frac{\overline{u_i u_j}}{k}\left(P^k - \varepsilon\right)$$

•Pressure-Strain Redistribution term:

$$\Phi_{ij} = \Phi_{ij,1} + \Phi_{ij,2} + \Phi'_{ij,1} + \Phi'_{ij,2}$$

where

$$\Phi_{ij,1} = -c_1 \frac{\varepsilon}{k}\left(\overline{u_i u_j} - \frac{2}{3}\delta_{ij} k\right)$$

$$\Phi_{ij,2} = -c_2\left(P_{ij} - \frac{2}{3}\delta_{ij} P^k\right), \quad P^k = \frac{1}{2} P_{\ell\ell}$$

$$\overline{u_i u_j} = \frac{2}{3}\delta_{ij} k + \frac{k}{\varepsilon} \frac{(1-c_2)\left(P_{ij} - \frac{2}{3}\delta_{ij} P\right) + \Phi'_{ij,1} + \Phi'_{ij,2}}{c_1 + P/\varepsilon - 1}$$

Explicit ASM (EASM or EARSM)

•Pope (1975) managed to derive an explicit expression for ASM in 2D:

$$\mathrm{b} = \mathrm{S}^* + \mathrm{S}^*\mathrm{W}^* - \mathrm{W}^*\mathrm{S}^* - 2\left\{(\mathrm{S}^*)^2 - \frac{1}{3}(\mathrm{S}^*)^2\mathrm{I}\right\}.$$

$$\mathrm{b} = \frac{3 - 2\eta_1 - 6\eta_2}{3}\mathrm{b}^*$$

$$\mathrm{S}^* = \frac{1}{2}\frac{k}{\varepsilon}\frac{(2 - C_3)S_{ij}}{\frac{1}{2}(C_1 + P_k/\varepsilon - 1)}$$

$$\mathrm{W}^* = \frac{1}{2}\frac{k}{\varepsilon}\frac{(2 - C_3)\omega_{ij}}{\frac{1}{2}(C_1 + P_k/\varepsilon - 1)}$$

$$\mathrm{b}^* = \frac{C_3 - 2}{C_2 - \frac{4}{3}}b_{ij}$$

$$b_{ij} = \frac{\overline{u_i u_j} - \frac{2}{3}k\delta_{ij}}{2k}$$

Physical modelling of the pressure-strain term

بصورت خود خوان مطالعه گردد. مراجع شامل مرجع ۷ ص۳۲۴ و جزوه داویدسون

روشهای پیش بینی تنشهای رینولدز $-\overline{u_i u_j}$

I. Algebraic models.

II. One-equation models.

III. Two-equation models.

IV. Reynolds stress models

Non-linear Eddy-viscosity Models

Boussinesq Assumption

Boussinesq assumption, where the *Reynolds stress tensor* in the time averaged Navier-Stokes equation is replaced by *the turbulent viscosity multiplied by the velocity gradients.*

$$\left[\mu(\bar{U}_{i,j} + \bar{U}_{j,i}) - \rho\overline{u_i u_j}\right]_{,j} = \left[(\mu + \mu_t)(\bar{U}_{i,j} + \bar{U}_{j,i})\right]_{,j}$$

which gives

$$\rho\overline{u_i u_j} = -\mu_t(\bar{U}_{i,j} + \bar{U}_{j,i}).$$

$$\rho\overline{u_i u_j} = -\mu_t(\bar{U}_{i,j} + \bar{U}_{j,i})$$

setting indices $i = j$

$$\overline{u_i u_i} \equiv 2k$$

where k is the turbulent kinetic energy

$$\overline{U}_{i,i} = 0$$

$$\overline{u_i u_i} \equiv 2k$$

$$\overline{u_i u_i} = -\mu_t(\overline{U}_{i,i} + \overline{U}_{i,i}) + ?$$

$$\rho\overline{u_i u_j} = -\mu_t(\bar{U}_{i,j} + \bar{U}_{j,i}) + \frac{2}{3}\delta_{ij}\rho k.$$

Note that contraction of δ_{ij} gives

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$$

Algebraic Models

In eddy viscosity models we want an expression for the turbulent viscosity $\mu_t = \rho\nu_t$. The dimension of ν_t is $[\mathrm{m}^2/\mathrm{s}]$ (same as ν). A turbulent velocity scale multiplied with a turbulent length scale gives the correct dimension,

$$\nu_t \propto \mathcal{U}\ell$$

$\mathcal{U}$ and l which are characteristic for the large turbulent scales

In an algebraic turbulence model the velocity gradient is used as a velocity scale and some physical length is used as the length scale

$$\nu_t = \ell_{mix}^2 \left| \frac{\partial U}{\partial y} \right|$$

where y is the coordinate normal to the wall, and where ℓ_{mix} is the mixing length, and the model is called the mixing length model

unknown and must be determined.

$$-\varrho \overline{u'v'} = \varrho \varepsilon_m (\partial u / \partial y)$$

ε_m like the kinematic viscosity ν.

$\varepsilon_m \sim$ length $\times$ velocity

mixing-length concept was first proposed by Prandtl

$$-\varrho \overline{u'v'} = \varrho l^2 \left| \frac{\partial u}{\partial y} \right| \frac{\partial u}{\partial y}$$

$$\varepsilon_m = l^2 \left| \frac{\partial u}{\partial y} \right|$$

According to von Kármán's hypothesis

$$l = \kappa \left| \frac{\partial u / \partial y}{\partial^2 u / \partial y^2} \right|$$

κ is an empirical constant known as von Kármán's constant.

$\varepsilon_m = C_\mu k^{1/2} l$ where l is a length scale, still related to the shear layer thickness,

turbulent kinetic energy

$$\varepsilon = \frac{k^{3/2}}{l}$$

rate of turbulence dissipation

$$c_\mu \equiv \frac{-\overline{u'v'}}{\frac{\partial u}{\partial y}} \frac{\varepsilon}{k^2}$$

$$-\varrho\overline{v'h'} = \varrho\varepsilon_h \frac{\partial h}{\partial y}$$

fluctuating enthalpy h'

$$-\varrho c_p \overline{T'v'} = \varrho c_p \varepsilon_h \frac{\partial T}{\partial y}$$

Sometimes it has been found to be convenient to introduce a "turbulent" Prandtl number Pr_t defined by

$$\mathrm{Pr}_t \equiv \frac{\varepsilon_m}{\varepsilon_h} = \frac{\overline{u'v'} / \frac{\partial u}{\partial y}}{\overline{T'v'} / \frac{\partial T}{\partial y}}$$

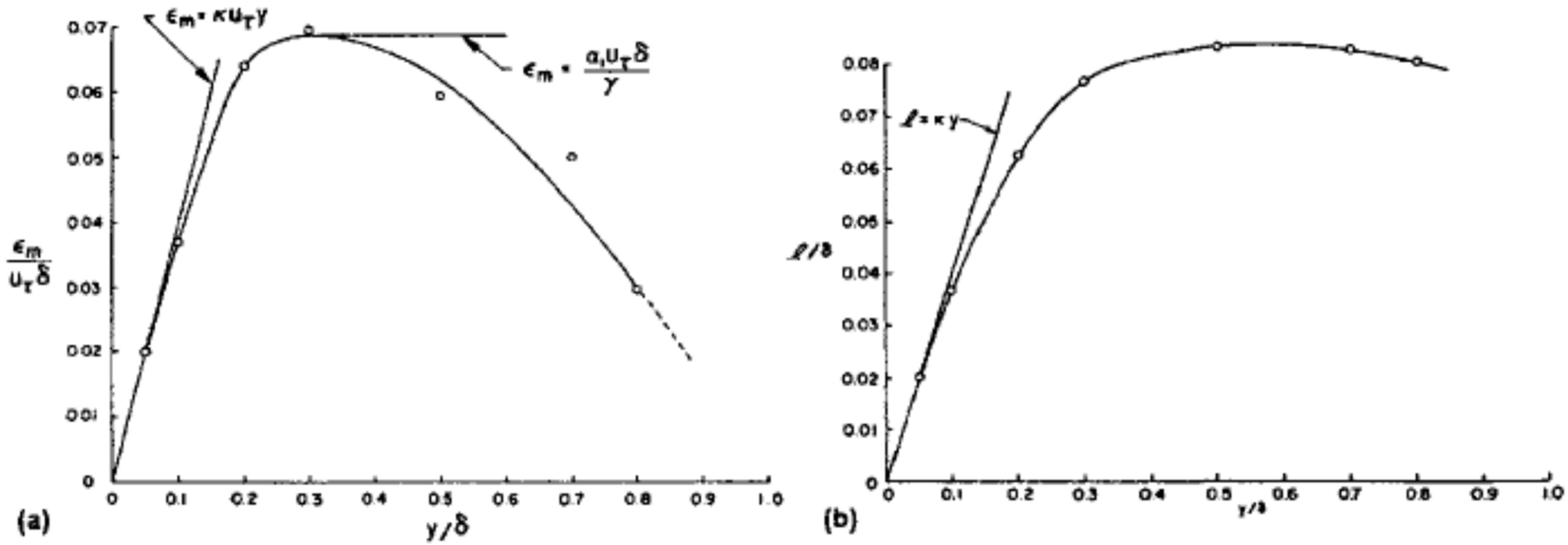

Fig. 4.9. Dimensionless (a) eddy-viscosity and (b) mixing-length distributions across a turbulent boundary layer at zero pressure gradient, according to the data of Klebanoff [5].

$$\varepsilon_m = \kappa u_\tau y$$

$$l = \kappa y\,,$$

κ is a universal constant 0.40–0.41

For high Reynolds numbers $l/r_0 = 0.14 - 0.08[1-(y/r_0)]^2 - 0.06[1-(y/r_0)]^4$

r_0 the radius of the pipe

Cebeci-Smith Model

$$\mu_T = \begin{cases} \mu_{T_i}, & y \le y_m \\ \mu_{T_o}, & y > y_m \end{cases}$$

where y_m is the smallest value of y for which $\mu_{T_i} = \mu_{T_o}$. The values of μ_T in the inner layer, μ_{T_i}, and the outer layer, μ_{T_o}, are computed as follows.

Inner Layer:

$$\mu_{T_i} = \rho \ell_{mix}^2 \left[\left(\frac{\partial U}{\partial y} \right)^2 + \left(\frac{\partial V}{\partial x} \right)^2 \right]^{1/2}$$

$$\ell_{mix} = \kappa y \left[1 - e^{-y^+/A^+} \right]$$

Outer layer:

$$\mu_{T_o} = \alpha \rho U_e \delta_v^* F_{Kleb}(y;\delta)$$

$\kappa = 0.40$, $\alpha = 0.0168$, $A^+ = 26 \left[1 + y \dfrac{dP/dx}{\rho u_\tau^2} \right]^{-1/2}$

F_{Kleb} is the Klebanoff intermittency function

$$F_{Kleb}(y;\delta) = \left[1 + 5.5 \left(\frac{y}{\delta} \right)^6 \right]^{-1}$$

$$\delta_v^* = \int_0^\delta (1 - U/U_e)\, dy$$

velocity thickness

U_e is boundary-layer edge velocity

Baldwin-Lomax Model

Inner Layer:

$$\mu_{T_i} = \rho \ell_{mix}^2 |\omega|$$

$$\ell_{mix} = \kappa y \left[1 - e^{-y^+/A_o^+}\right]$$

Outer Layer:

$$\mu_{T_o} = \rho \alpha C_{cp} F_{wake} F_{Kleb}(y; y_{max}/C_{Kleb})$$

$$F_{wake} = \min\left[y_{max} F_{max}; C_{wk} y_{max} U_{dif}^2 / F_{max}\right]$$

$$F_{max} = \frac{1}{\kappa}\left[\max_y(\ell_{mix}|\omega|)\right]$$

where y_{max} is the value of y at which $\ell_{mix}|\omega|$ achieves its maximum value.

$$\left.\begin{array}{lll} \kappa = 0.40, & \alpha = 0.0168, & A_o^+ = 26 \\ C_{cp} = 1.6, & C_{Kleb} = 0.3, & C_{wk} = 1 \end{array}\right\}$$

$$F_{Kleb}(y;\delta) = \left[1 + 5.5\left(\frac{y}{\delta}\right)^6\right]^{-1}$$

δ replaced by y_{max}/C_{Kleb}

$$\omega = \left[\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z}\right)^2 + \left(\frac{\partial U}{\partial z} - \frac{\partial W}{\partial x}\right)^2\right]^{1/2}$$

ω is the magnitude of the vorticity

Equations for Kinetic Energy

Navier-Stokes equation which reads assuming steady, incompressible

$$(\rho U_i U_j)_{,j} = -P_{,i} + \mu U_{i,jj} \quad (1)$$

The time averaged Navier-Stokes equation

$$(\rho \bar{U}_i \bar{U}_j)_{,j} = -\bar{P}_{,i} + \mu \bar{U}_{i,jj} - \rho(\overline{u_i u_j})_{,j} \quad (2)$$

(1) - (2) $\times u_i$ →

$$\overline{\left[\rho U_i U_j - \rho \bar{U}_i \bar{U}_j\right]_{,j} u_i} = -\overline{\left[P - \bar{P}\right]_{,i} u_i} + \mu \overline{\left[U_i - \bar{U}_i\right]_{,jj} u_i} + \rho \overline{(\overline{u_i u_j})_{,j} u_i}$$

The left-hand side can be rewritten as

$$\rho \overline{\left[(\bar{U}_i + u_i)(\bar{U}_j + u_j) - \bar{U}_i \bar{U}_j\right]_{,j} u_i} = \rho \overline{\left[\bar{U}_i u_j + u_i \bar{U}_j + u_i u_j\right]_{,j} u_i}.$$

Using the continuity equation $(\rho u_j)_{,j} = 0$.

$$\overline{\left(\rho \bar{U}_i u_j\right)_{,j} u_i} = \rho \overline{u_i u_j} \bar{U}_{i,j}$$

$$\rho\overline{[\bar{U}_i u_j + u_i\bar{U}_j + u_i u_j]_{,j}\, u_i}.$$

$$\frac{1}{2}\overline{(\rho u_j u_i u_i)}_{,j}$$

the second term
$(\rho\bar{U}_j)_{,j} = 0$

$$\overline{(\rho\bar{U}_j k)}_{,j} = \bar{U}_j\rho\overline{\left[\frac{1}{2}u_i u_i\right]}_{,j} = \frac{1}{2}\rho\bar{U}_j\left\{\overline{u_i u_{i,j} + u_i u_{i,j}}\right\} = \overline{u_i\left(\rho\bar{U}_j u_i\right)_{,j}}$$

$$\overline{[\rho U_i U_j - \rho\bar{U}_i\bar{U}_j]_{,j}\, u_i} =$$

$$-\overline{[P-\bar{P}]_{,i}\, u_i} + \mu\overline{[U_i-\bar{U}_i]_{,jj}\, u_i} + \overline{\rho(\overline{u_i u_j})_{,j} u_i}$$

0

$$-\overline{p_{,i}u_i} = -(\overline{pu_i})_{,i}$$

$$\mu\overline{u_{i,jj}u_i} = \mu\left\{\overline{(u_{i,j}u_i)_{,j} - u_{i,j}u_{i,j}}\right\}$$

$$\mu\overline{(u_{i,j}u_i)}_{,j} = \mu\frac{1}{2}\overline{(u_i u_i)}_{,jj} = \mu k_{,jj}$$

The Equation for $1/2(U_i U_i)$

instantaneous kinetic energy $K = \frac{1}{2}U_i U_i$

Navier-Stokes equation. Multiply by U_i → $U_i(\rho U_i U_j)_{,j} = -U_i P_{,i} + \mu U_i U_{i,jj}$.

$$U_i(\rho U_i U_j)_{,j} = (\rho U_i U_i U_j)_{,j} - \rho U_i U_j U_{i,j} = \rho U_j (U_i U_i)_{,j} - \frac{1}{2}\rho U_j (U_i U_i)_{,j}$$

$$= \frac{1}{2}\rho U_j (U_i U_i)_{,j} = (\rho U_j K)_{,j}$$

$$-U_i P_{,i} = -(U_i P)_{,i}.$$

$$\mu U_i U_{i,jj} = \mu K_{,jj} - \mu U_{i,j} U_{i,j}.$$

$$(\rho U_j K)_{,j} = \mu K_{,jj} - (U_i P)_{,i} - \mu U_{i,j} U_{i,j}$$

Viscous diffusion **Dissipation**

the transport equation for the turbulent kinetic energy.

$$\underbrace{(\rho \bar{U}_j k)_{,j}}_{I} = \underbrace{-\rho \overline{u_i u_j} \bar{U}_{i,j}}_{II} - \underbrace{\left[\overline{u_j p} + \frac{1}{2} \rho \overline{u_j u_i u_i} - \mu k_{,j} \right]_{,j}}_{III} - \underbrace{\mu \overline{u_{i,j} u_{i,j}}}_{IV}$$

Convection **Production >0** **Dissipation < 0**

III. The two first terms represent **turbulent diffusion** by pressure-velocity fluctuations, and velocity fluctuations, respectively. The last term is viscous diffusion.

In boundary-layer flow the exact k equation

$$\frac{\partial \rho \bar{U} k}{\partial x} + \frac{\partial \rho \bar{V} k}{\partial y} = -\rho \overline{uv} \frac{\partial \bar{U}}{\partial y} - \frac{\partial}{\partial y} \left[\overline{pv} + \frac{1}{2} \rho \overline{v u_i u_i} - \mu \frac{\partial k}{\partial y} \right] - \mu \overline{u_{i,j} u_{i,j}}$$

$\partial u_i / \partial x \ll \partial u_i / \partial y$ is not valid.

The Equation for $1/2(\bar{U}_i\bar{U}_i)$

the time-averaged Navier-Stokes equations multiply $\bar{U}_i$

$$\bar{U}_i(\rho\bar{U}_i\bar{U}_j)_{,j} = -\bar{U}_i\bar{P}_{,i} + \mu\bar{U}_i\bar{U}_{i,jj} - \bar{U}_i\rho(\overline{u_iu_j})_{,j}$$

$$\mu\overline{U_{i,j}U_{i,j}} = \mu\bar{U}_{i,j}\bar{U}_{i,j} + \mu\overline{u_{i,j}u_{i,j}}.$$

As the scales of $\bar{U}$ is much larger than those of u_i, i.e. $|u_{i,j}| \gg |U_{i,j}|$

$$\mu\overline{U_{i,j}U_{i,j}} \simeq \mu\overline{u_{i,j}u_{i,j}}.$$

$$(\rho\bar{U}_j\bar{K})_{,j} = \mu\bar{K}_{,jj} - (\bar{U}_i\bar{P})_{,i} - \mu\bar{U}_{i,j}\bar{U}_{i,j} - \bar{U}_i\rho(\overline{u_iu_j})_{,j}.$$

$$-\bar{U}_i\rho(\overline{u_iu_j})_{,j} = -(\bar{U}_i\rho\overline{u_iu_j})_{,j} + \rho(\overline{u_iu_j})\bar{U}_{i,j}$$

velocity-stress interaction

$$(\rho\bar{U}_j\bar{K})_{,j} = \mu\bar{K}_{,jj} - (\bar{U}_i\bar{P})_{,i} - \mu\bar{U}_{i,j}\bar{U}_{i,j} - (\bar{U}_i\rho\overline{u_iu_j})_{,j} + \rho\overline{u_iu_j}\bar{U}_{i,j}$$

Viscous diffusion

Viscous dissipation

loss of energy to the fluctuating velocity field

$$\underbrace{(\rho \bar{U}_j k)_{,j}} = \underbrace{-\rho \overline{u_i u_j} \bar{U}_{i,j}} - \left[\overline{u_j p} + \frac{1}{2} \rho \overline{u_j u_i u_i} - \mu k_{,j} \right]_{,j} \underbrace{- \mu \overline{u_{i,j} u_{i,j}}}$$

$$(\rho U_j K)_{,j} = \mu K_{,jj} - (U_i P)_{,i} - \mu U_{i,j} U_{i,j}$$

$$(\rho \bar{U}_j \bar{K})_{,j} = \mu \bar{K}_{,jj} - (\bar{U}_i \bar{P})_{,i} - \mu \bar{U}_{i,j} \bar{U}_{i,j} - (\bar{U}_i \rho \overline{u_i u_j})_{,j} + \rho \overline{u_i u_j} \bar{U}_{i,j}$$

Dissipation $-\mu U_{i,j} U_{i,j}$ **Time-averaged** $\rightarrow$ $-\mu \overline{U_{i,j} U_{i,j}} = -\mu \bar{U}_{i,j} \bar{U}_{i,j} - \mu \overline{u_{i,j} u_{i,j}}.$

In the $\bar{K}$ equation the dissipation term and the negative production term (representing loss of kinetic energy to the k field)

$$-\mu \bar{U}_{i,j} \bar{U}_{i,j} + \rho \overline{u_i u_j} \bar{U}_{i,j}$$

the k equation the production and the dissipation terms $-\rho \overline{u_i u_j} \bar{U}_{i,j} - \mu \overline{u_{i,j} u_{i,j}}$

ترم تلفات در معادله لحظه ای انرژی در متوسط میدان سیال و میدان نوسانات توزیع می گردد. با توجه با ناچیز بودن ترم تلفات در متوسط میدان , قسمت اعظم انرزی توسط نوسانات تلف می گردد

$$\underbrace{(\rho \bar{U}_j k)_{,j}} = \underbrace{-\rho \overline{u_i u_j} \bar{U}_{i,j}} - \left[\overline{u_j p} + \frac{1}{2} \rho \overline{u_j u_i u_i} - \mu k_{,j} \right]_{,j} - \underbrace{\mu \overline{u_{i,j} u_{i,j}}}$$

$$\rho \overline{u_i u_j} = -\mu_t (\bar{U}_{i,j} + \bar{U}_{j,i}) + \frac{2}{3} \delta_{ij} \rho k. \Rightarrow P_k = -\rho \overline{u_i u_j} \bar{U}_{i,j} = \mu_t \left(\bar{U}_{i,j} + \bar{U}_{j,i} \right) \bar{U}_{i,j} - \frac{2}{3} \rho k \bar{U}_{i,i}$$

0

using a gradient law where we assume that k is diffused *down* the gradient, i.e from region of high k to regions of small k (cf. Fourier's law for heat flux: heat is diffused from hot to cold regions)

$$\frac{1}{2} \rho \overline{u_j u_i u_i} = -\frac{\mu_t}{\sigma_k} k_{,j}$$

where σ_k is the turbulent Prandtl number for k.

pressure diffusion term is small thus it is simply neglected.

$$\left[\overline{u_j p} + \frac{1}{2} \rho \overline{u_j u_i u_i} - \mu k_{,j} \right]_{,j} \Rightarrow \left[-\frac{\mu_t}{\sigma_k} k_{,j} - \mu k_{,j} \right]_{,j}$$

$$\varepsilon = \mathcal{O}\left(\frac{\mathcal{U}^2}{\ell/\mathcal{U}}\right) = \mathcal{O}\left(\frac{\mathcal{U}^3}{\ell}\right)$$

$$\mathcal{U} = \sqrt{k}$$

$$\varepsilon \equiv \nu\overline{u_{i,j}u_{i,j}} = \frac{k^{\frac{3}{2}}}{\ell}$$

$$\epsilon = C_D k^{3/2}/\ell$$

$$0.07 < C_D < 0.09$$

$$(\rho\bar{U}_j k)_{,j} = \left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)k_{,j}\right]_{,j} + P_k - \rho\frac{k^{\frac{3}{2}}}{\ell}$$

? ?

For boundary-layer flow

$$C_D\rho\frac{k^{3/2}}{\ell}$$

$$\frac{\partial\rho\bar{U}k}{\partial x} + \frac{\partial\rho\bar{V}k}{\partial y} = \frac{\partial}{\partial y}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] + \mu_t\left(\frac{\partial\bar{U}}{\partial y}\right)^2 - \rho\frac{k^{\frac{3}{2}}}{\ell}.$$

The main *disadvantage* of this type of model is that it is not applicable to general flows since it is *not possible to find a general expression for an algebraic length scale.*

Two-Equation Turbulence Models

The Modelled ε Equation

$$\mathcal{N}(u_i) = \rho\frac{\partial u_i}{\partial t} + \rho u_k\frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} - \mu\frac{\partial^2 u_i}{\partial x_k \partial x_k}$$

Navier-Stokes equation $\mathcal{N}(u_i) = 0$

$$\overline{2\nu\frac{\partial u_i'}{\partial x_j}\frac{\partial}{\partial x_j}\left[\mathcal{N}(u_i)\right]} = 0$$

Production of Dissipation

$$\rho\frac{\partial \epsilon}{\partial t} + \rho U_j\frac{\partial \epsilon}{\partial x_j} = -2\mu\left[\overline{u'_{i,k}u'_{j,k}} + \overline{u'_{k,i}u'_{k,j}}\right]\frac{\partial U_i}{\partial x_j} - 2\mu\,\overline{u'_k u'_{i,j}}\frac{\partial^2 U_i}{\partial x_k \partial x_j}$$

Dissipation of Dissipation,

$$-2\mu\,\overline{u'_{i,k}u'_{i,m}u'_{k,m}} - 2\mu\,\nu\,\overline{u'_{i,km}u'_{i,km}}$$

$$+\frac{\partial}{\partial x_j}\left[\mu\frac{\partial \epsilon}{\partial x_j} - \mu\,\overline{u'_j u'_{i,m}u'_{i,m}} - 2\nu\,\overline{p'_{,m}u'_{j,m}}\right]$$

Molecular Diffusion of Dissipation and Turbulent Transport of Dissipation,

$$P_\varepsilon = -c_{\varepsilon 1}\frac{\varepsilon}{k}\left(\bar{U}_{i,j}+\bar{U}_{j,i}\right)\bar{U}_{i,j}$$

$$\text{diss.term} = -c_{\varepsilon 2}\rho\frac{\varepsilon^2}{k}.$$

Note that for the production term we have $P_\varepsilon = c_{\varepsilon 1}(\varepsilon/k)P_k$

$$(\rho\bar{U}_j\varepsilon)_{,j} = \left[\left(\mu+\frac{\mu_t}{\sigma_\varepsilon}\right)\varepsilon_{,j}\right]_{,j} + \frac{\varepsilon}{k}\left(c_{\varepsilon 1}P_k - c_{\varepsilon 2}\rho\varepsilon\right)$$

For boundary-layer flow

$$\frac{\partial\rho\bar{U}\varepsilon}{\partial x}+\frac{\partial\rho\bar{V}\varepsilon}{\partial y} = \frac{\partial}{\partial y}\left[\left(\mu+\frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial\varepsilon}{\partial y}\right] + c_{\varepsilon 1}\frac{\varepsilon}{k}\mu_t\left(\frac{\partial\bar{U}}{\partial y}\right)^2 - \rho c_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

Eddy Viscosity

$$\mu_T = \rho C_\mu k^2/\epsilon$$

Turbulence Kinetic Energy

$$\rho\frac{\partial k}{\partial t} + \rho U_j\frac{\partial k}{\partial x_j} = \tau_{ij}\frac{\partial U_i}{\partial x_j} - \rho\epsilon + \frac{\partial}{\partial x_j}\left[(\mu + \mu_T/\sigma_k)\frac{\partial k}{\partial x_j}\right]$$

Dissipation Rate

$$\rho\frac{\partial \epsilon}{\partial t} + \rho U_j\frac{\partial \epsilon}{\partial x_j} = C_{\epsilon 1}\frac{\epsilon}{k}\tau_{ij}\frac{\partial U_i}{\partial x_j} - C_{\epsilon 2}\rho\frac{\epsilon^2}{k} + \frac{\partial}{\partial x_j}\left[(\mu + \mu_T/\sigma_\epsilon)\frac{\partial \epsilon}{\partial x_j}\right]$$

Closure Coefficients

$$C_{\epsilon 1} = 1.44, \quad C_{\epsilon 2} = 1.92, \quad C_\mu = 0.09, \quad \sigma_k = 1.0, \quad \sigma_\epsilon = 1.3$$

Auxiliary Relations

$$\omega = \epsilon/(C_\mu k) \quad \text{and} \quad \ell = C_\mu k^{3/2}/\epsilon$$