$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\overline{u_i u_j}) = -\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)$$

non-linear term $\overline{u_i u_j}$

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \quad ,$$

$$u' = u - \overline{u}$$

Leonard expresses the non-linear term in the form of a triple summation:

$$\overline{u_i u_j} = \overline{(\overline{u}_i + u'_i)(\overline{u}_j + u'_j)} = \overline{\overline{u}_i \overline{u}_j} + \overline{\overline{u}_i u'_j} + \overline{\overline{u}_j u'_i} + \overline{u'_i u'_j}$$

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\overline{\overline{u}_i \overline{u}_j}) = -\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) - \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\tau_{ij} = C_{ij} + R_{ij} = \overline{u_i u_j} - \overline{\overline{u}_i \overline{u}_j}$$

$$C_{ij} = \overline{\overline{u}_i u'_j} + \overline{\overline{u}_j u'_i}$$

$$R_{ij} = \overline{u'_i u'_j} \quad .$$

cross-stress tensor

represents the interactions between large and small scales

Reynolds subgrid tensor

reflects the interactions between subgrid scales

$$\overline{\overline{u}_i \overline{u}_j} = \left(\overline{\overline{u}_i \overline{u}_j} - \overline{u}_i \overline{u}_j\right) + \overline{u}_i \overline{u}_j$$
$$= L_{ij} + \overline{u}_i \overline{u}_j \quad .$$

The new *L* term, called Leonard tensor, represents interactions among the large scales

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{u}_i \overline{u}_j\right) = -\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial}{\partial x_j}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) - \frac{\partial \tau_{ij}}{\partial x_j}$$

The subgrid tensor τ → $\tau_{ij} = L_{ij} + C_{ij} + R_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j$

$\overline{u'} \neq 0$

$\overline{\overline{u}}_i \neq \overline{u}_i$

This is true for box filters. Note that for the spectral cut-off filter $\overline{\overline{u}}_i = \overline{u}_i$

However, in finite volume methods, box filters are always used.

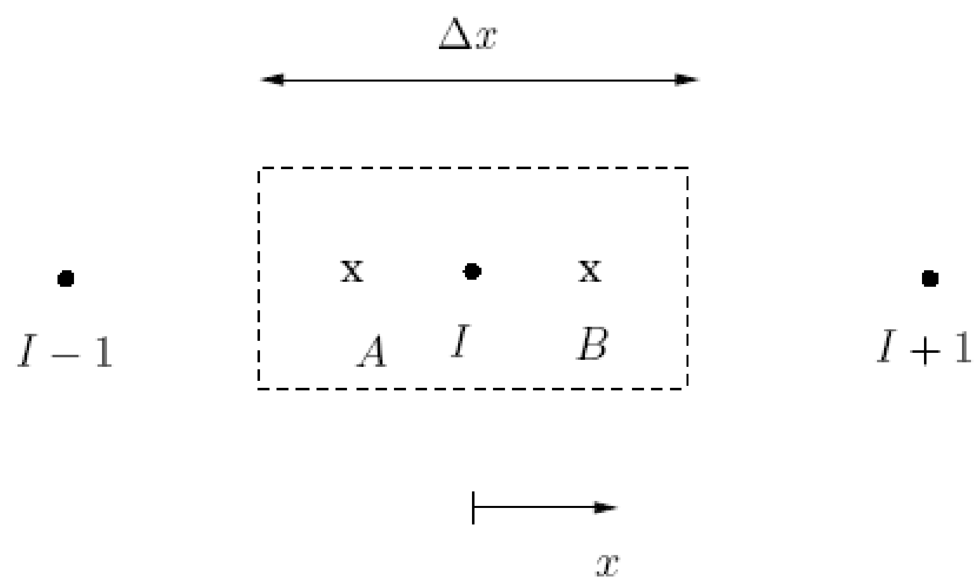
Differences between time-averaging (RANS) and space filtering (LES)

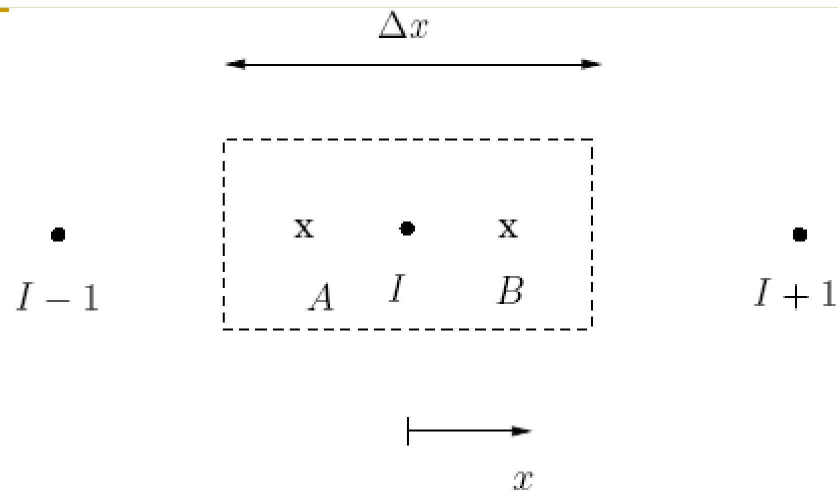
$$\langle\langle u \rangle\rangle = \frac{1}{2T}\int_{-T}^{T} \langle u \rangle dt = \frac{1}{2T}\langle u \rangle 2T = \langle u \rangle$$

<u>In **LES**</u>, $\overline{\overline{u}} \neq \overline{u}$

(and since $u = \overline{u} + u''$ we get $\overline{u''} \neq 0$).

$$u' = u''$$





$$\overline{u}_I = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \bar{u}(\xi)d\xi = \frac{1}{\Delta x}\left(\int_{-\Delta x/2}^{0} \bar{u}(\xi)d\xi + \int_{0}^{\Delta x/2} \bar{u}(\xi)d\xi\right) =$$
$$= \frac{1}{\Delta x}\left(\frac{\Delta x}{2}\bar{u}_A + \frac{\Delta x}{2}\bar{u}_B\right).$$

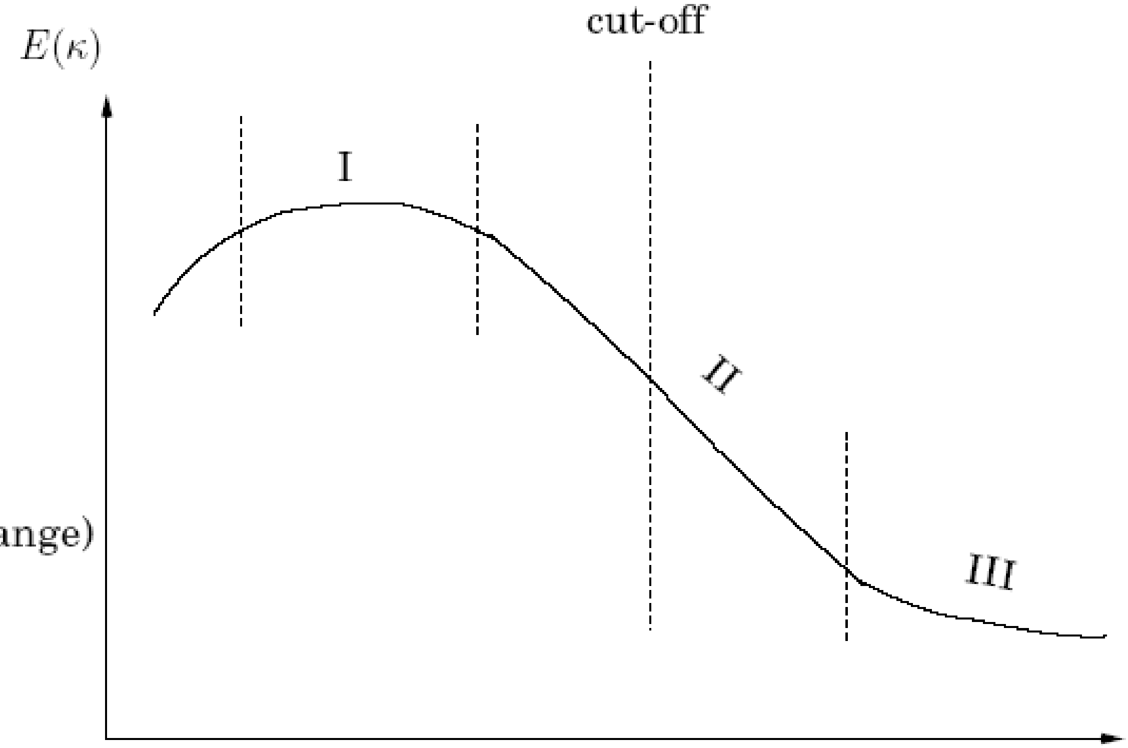
The trapezoidal rule, which is second-order accurate, was used to estimate the integrals. $\bar{u}$ at locations A and B (see figure above) is estimated by linear interpolation, which gives

$$\bar{u}_I = \frac{1}{2}\left[\left(\frac{1}{4}\bar{u}_{I-1} + \frac{3}{4}\bar{u}_I\right) + \left(\frac{3}{4}\bar{u}_I + \frac{1}{4}\bar{u}_{I+1}\right)\right]$$
$$= \frac{1}{8}\left(\bar{u}_{I-1} + 6\bar{u}_I + \bar{u}_{I+1}\right) \neq \bar{u}_I$$

Resolved & SGS scales

• The basic idea in LES is to resolve (large) grid scales (GS), and to model (small) subgrid-scales (SGS).

The limit (cut-off) between GS and SGS is supposed to take place in the inertial subrange (II),



I: large, energy-containing scales
II: inertial subrange (Kolmogorov $-5/3$-range)
III: dissipation subrange

Subgrid model

$$\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j$$

The simplest model is the Smagorinsky model (Smagorinsky,1963):

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -2\nu_{sgs}\bar{s}_{ij}$$

$$\Delta = (\Delta x_1 \times \Delta x_2 \times \Delta x_3)^{1/3}$$

$$\nu_{sgs} = (C_S\Delta)^2 \sqrt{2\bar{s}_{ij}\bar{s}_{ij}} \equiv (C_S\Delta)^2 |\bar{s}|$$

$$\bar{s}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

Because the SGS turbulent fluctuations near a wall go to zero, so must SGS viscosity. A damping function f_μ is added to ensure this

$$f_\mu = 1 - \exp(-y^+/26) \rightarrow \nu_t = \left(C_S\Delta\left\{1 - exp\left(\frac{-y^+}{26}\right)\right\}\right)^2 \sqrt{2\bar{S}_{ij}\bar{S}_{ij}}$$

Disadvantage of Smagorinsky model: the "constant" C_S is not constant, but it is flow-dependent. It is found to vary in the range from $C = 0.065$ (Moin & Kim, 1982) to $C = 0.25$ (Jones & Wille, 1995).

mixing length theory → $\nu_t = \ell^2 \left| \frac{\partial U}{\partial y} \right|$

Generalized to three dimensions → $\nu_t = \ell^2 \left[\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j} \right]^{1/2} = \ell^2 \left(2 S_{ij} S_{ij} \right)^{1/2} \equiv \ell^2 |S|$

In the Smagorinsky model the SGS turbulent length scale corresponds to $\ell = C_S \Delta$ so that

$$\nu_{sgs} = (C_S \Delta)^2 |\bar{s}|$$

Smagorinsky model derived from the $k^{-5/3}$ law

در صورتی که نرخ تولید انرژی و اتلاف آن با هم برابر باشد

$\ell = C_S \Delta$

$\varepsilon \approx u^3 / l$ l طول انتگرال و u سرعت ادیهای بزرگ

$\varepsilon \approx u'^3 / \Delta$ Δ اندازه بزرگترین ادی کوچک

$\nu_{sgs} = \varepsilon^a (C_S \Delta)^b$ Dimensional analysis yields $a = 1/3, b = 4/3$ so that

$$\nu_{sgs} = (C_S \Delta)^{4/3} \varepsilon^{1/3}.$$

The scales in the inertial subrange are isotropic, and thus they are in local equilibrium, i.e. in the k_{sgs} equation we have that production balances dissipation

$$\left.\begin{aligned} 2\nu_{sgs}\bar{s}_{ij}\bar{s}_{ij} &= \varepsilon. \\ \nu_{sgs} &= (C_S\Delta)^{4/3}\varepsilon^{1/3} \end{aligned}\right\} \quad \left.\begin{aligned} \nu_{sgs}^3 &= (C_S\Delta)^4\varepsilon = (C_S\Delta)^4\nu_{sgs}(2\bar{s}_{ij}\bar{s}_{ij}) \\ \nu_{sgs} &= (C_S\Delta)^2|\bar{s}| \\ |\bar{s}| &= (2\bar{s}_{ij}\bar{s}_{ij})^{1/2} \end{aligned}\right\} \quad \varepsilon = (C_S\Delta)^2|\bar{s}|^3$$

Let's compute $|\bar{s}|^2$ in the Kolmogorov $-5/3$ range, where the turbulence is isotropic and homogeneous. We can write

$$|\bar{s}|^2 = 2\bar{s}_{ij}\bar{s}_{ij} = \frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \bar{u}_i}{\partial x_j}$$ is valid because the turbulence is isotropic

$$= -\left.\frac{\partial^2 Q_{i,i}}{\partial r_j \partial r_j}\right|_{r=0}$$ is due to that the turbulence is homogeneous

The general two-point correlation $Q_{i,j}$ of u_i and u_j can be expressed by the energy spectrum tensor as:

$$Q_{i,j}(\mathbf{r}) = \int_0^\infty E_{i,j}(\kappa)\exp(\imath\kappa_m r_m)d\kappa_1 d\kappa_2 d\kappa_3$$

Taking the trace and derivating twice gives

$$\frac{\partial^2 Q_{i,i}}{\partial r_j \partial r_j} = -\int_0^\infty \kappa^2 E_{i,i}(\kappa)\exp(\imath\kappa_m r_m)d\kappa_1 d\kappa_2 d\kappa_3$$

Integrating over a spherical shell yields:

$$\frac{\partial^2 Q_{i,i}}{\partial r_j \partial r_j} = -4\pi\int_0^\infty \kappa^4 E_{i,i}(\kappa)\frac{\sin(\kappa r)}{\kappa r}d\kappa$$

This determines the kinetic-energy spectrum, density of kinetic energy at wavenumber *k*, and such that

$$E(k,t) = 2\pi k^2 \hat{U}(k,t) \text{ in three dimensions,}$$

$$E(\kappa) = 2\pi\kappa^2 E_{i,i}(\kappa)$$

letting $r \to 0$

$$|\bar{s}|^2 = 2\int_0^\infty \kappa^2 \bar{E}(\kappa) d\kappa \qquad \bar{E}(\kappa) = \hat{\bar{u}}^2(\kappa))$$

This is the *filtered* spectrum.

Introduce the filtering function Ĝ for the cut-off filter: $|\bar{s}|^2 = 2\int_0^\infty \kappa^2 \hat{G}^2(\kappa) E(\kappa) d\kappa$

Using Kolmogorov spectrum $E(\kappa) = C\kappa^{-5/3}\varepsilon^{2/3}$

$$|\bar{s}|^2 = 2\int_0^{\kappa_c} \kappa^2 C\kappa^{-5/3}\varepsilon^{2/3} d\kappa = \frac{3}{2}\pi^{4/3} C\varepsilon^{2/3}\Delta^{-4/3}$$

$$\varepsilon = (C_S\Delta)^2 |\bar{s}|^3$$

$$\varepsilon = (C_S\Delta)^2 \left(\frac{3}{2}C\right)^{3/2} \pi^2 \varepsilon \Delta^{-2}$$

$$C_S = \frac{1}{\pi}\left(\frac{2}{3C}\right)^{3/4} = 0.17$$

For more detail see Ref. [8]

Kolmogorov constant $C = 1.5$

Structure Function Model

در سال ۱۹۹۲ توسط متایس و لزیربر اساس مدل (Spectral Eddy Viscosity)

$$\nu_t(\mathbf{x}, \Delta x) = \frac{2}{3} C_K^{-3/2} \left[\frac{E_{\mathbf{x}}(k_C)}{k_C}\right]^{1/2}$$

$$F_2(\mathbf{x}, \Delta x) = 4\int_0^{k_C} E(k)\left[1 - \frac{\sin(k\Delta x)}{k\Delta x}\right] dk$$

$$F_2(\mathbf{x}, \Delta x) = \left\langle \left\| \overline{\mathbf{u}}(\mathbf{x}, t) - \overline{\mathbf{u}}(\mathbf{x} + \Delta x, t) \right\|^2 \right\rangle_{\Delta x}$$

$$\nu_t(\mathbf{x},\Delta x) = \frac{2}{3} C_K^{-3/2} \Delta x \left[F_2(\mathbf{x},\Delta x)\right]^{1/2}$$

$$C_K \cong ۴/۱$$

$$\nu_t(\mathbf{x},\Delta x) = 0.063\,\Delta x \left[F_2(\mathbf{x},\Delta x)\right]^{1/2}$$

برای شبکه غیر یکنواخت متعامد و استفاده از مدل ۶ نقطه ای

$$\nu_t^{SF}(\mathbf{x},\Delta c) = 0.063\,\Delta c \sqrt{F_2(\mathbf{x},\Delta c)}$$

$$\Delta c = (\Delta x_1 \times \Delta x_2 \times \Delta x_3)^{1/3}$$

$$F_2(\mathbf{x},\Delta c) = \frac{1}{6}\sum_{i=1}^{3} \left\langle \left[\bar{\mathbf{u}}(\mathbf{x},t) - \bar{\mathbf{u}}(\mathbf{x}+\Delta x_i,t)\right]^2 + \left[\bar{\mathbf{u}}(\mathbf{x},t) - \bar{\mathbf{u}}(\mathbf{x}-\Delta x_i,t)\right]^2 \right\rangle \left(\frac{\Delta c}{\Delta x_i}\right)^{2/3}$$

مدل (SSF) *Selective Structure Function* در سال ۱۹۹۳ توسط دیوید

ایده اصلی در این روش بر اساس حذف اثر لزجت متلاطم در قسمتی از جریان که به اندازه کافی سه بعدی نمی باشد استوار است

$$\nu_t^{SSF}(\mathbf{x},\Delta c) = 0.1638\Phi_{20^\circ}(\mathbf{x},t)C_K^{-3/2}\Delta c[F_2(\mathbf{x},\Delta c)]^{1/2}$$

$$\Phi_{20^\circ}(\mathbf{x},t) = \begin{cases} 1 & if \quad \alpha \geq 20^\circ \\ 0 & if \quad \alpha \prec 20^\circ \end{cases}$$

تابع ترکیبی

$$\Phi'_{20^\circ}(\mathbf{x},t) = \begin{cases} 0 & for \quad \alpha \prec 10^\circ \\ e^{-\left(\frac{d\alpha}{3^\circ}\right)^2} & for \quad 20^\circ \geq \alpha \geq 10^\circ \quad and \quad d\alpha = \left|\alpha - 20^\circ\right| \\ 1 & for \quad \alpha \succ 20^\circ \end{cases}$$

Modified Selective Structure Function

در سال ۲۰۰۱ آکرمن و متایس

$$\nu_t^{MSSF}(\mathbf{x},\Delta c) = C_{MSSF}\Phi_{\alpha_c}(\mathbf{x},t)C_K^{-3/2}\Delta c[F_2(\mathbf{x},\Delta c)]^{1/2}$$

C_{MSSF} برابر ۱۴۲/۰

$$\Phi_{\alpha_c}(x,t) = \begin{cases} 1 & if \quad \alpha \succ \alpha_c \\ 0 & if \quad \alpha \prec \alpha_c \end{cases}$$

$$\alpha_C\left(\frac{k_C}{k_i}\right) = \begin{cases} 23\left(\frac{k_C}{k_i}\right)^{-0.4} & for \quad k_C/k_i \le 10 \\ 9 & for \quad k_C/k_i > 10 \end{cases}$$

The dynamic model

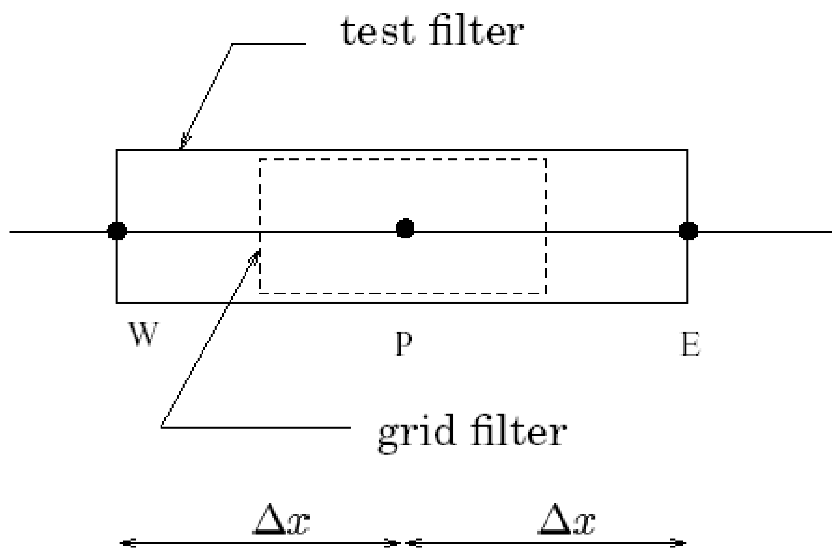
In this model of Germano *et al.* (1991) the constant C is not arbitrarily chosen (or optimized), but it is computed.

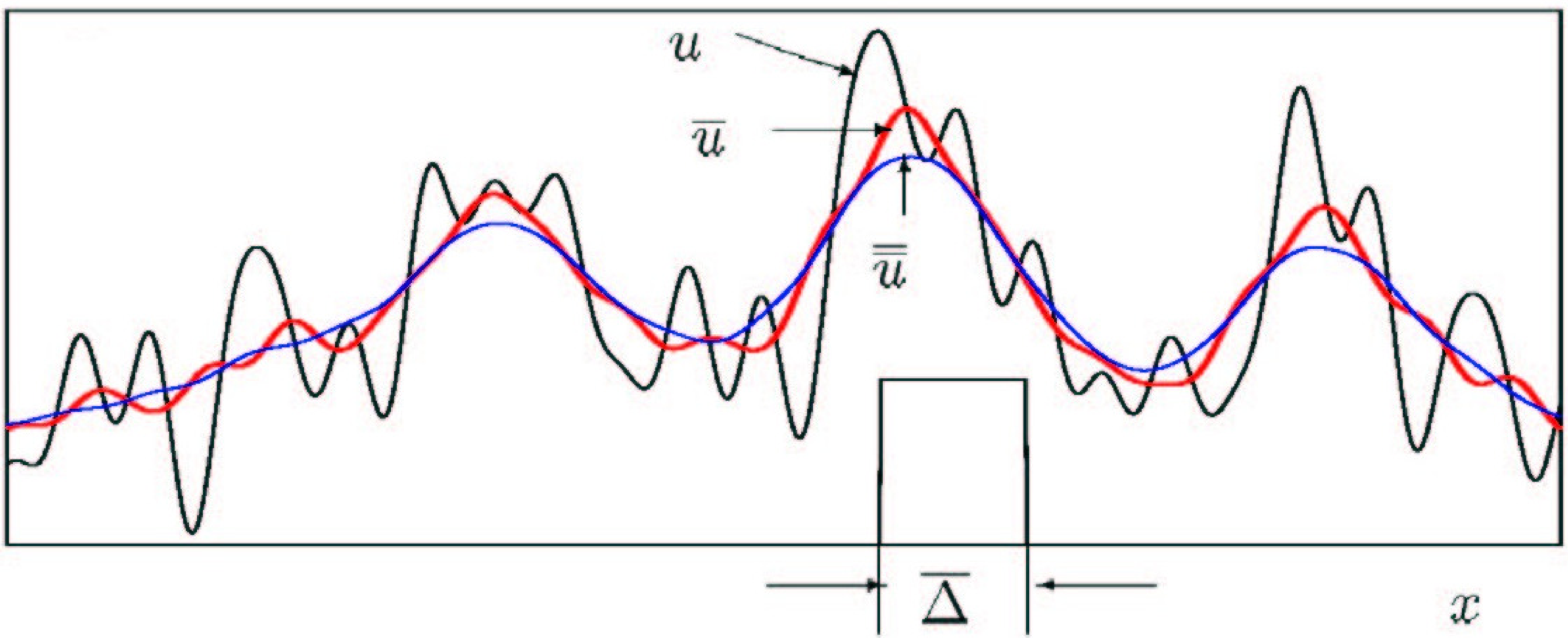
If we apply two filters to Navier-Stokes [grid filter and a second, coarser filter (test filter, denoted by $\widehat{\cdot}$)] where $\widehat{\Delta} = 2\Delta$ we get

$$\frac{\partial \widehat{\bar{u}}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\widehat{\bar{u}}_i \widehat{\bar{u}}_j\right) = -\frac{1}{\rho}\frac{\partial \widehat{\bar{p}}}{\partial x_i} + \nu \frac{\partial^2 \widehat{\bar{u}}_i}{\partial x_j \partial x_j} - \frac{\partial T_{ij}}{\partial x_j}$$

where the subgrid stresses on the test level now are given by

$$T_{ij} = \widehat{\overline{u_i u_j}} - \widehat{\bar{u}}_i \widehat{\bar{u}}_j$$


test filter
W
P
E
grid filter
Δx
Δx


u
ū
x

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_i \bar{u}_j) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}$$

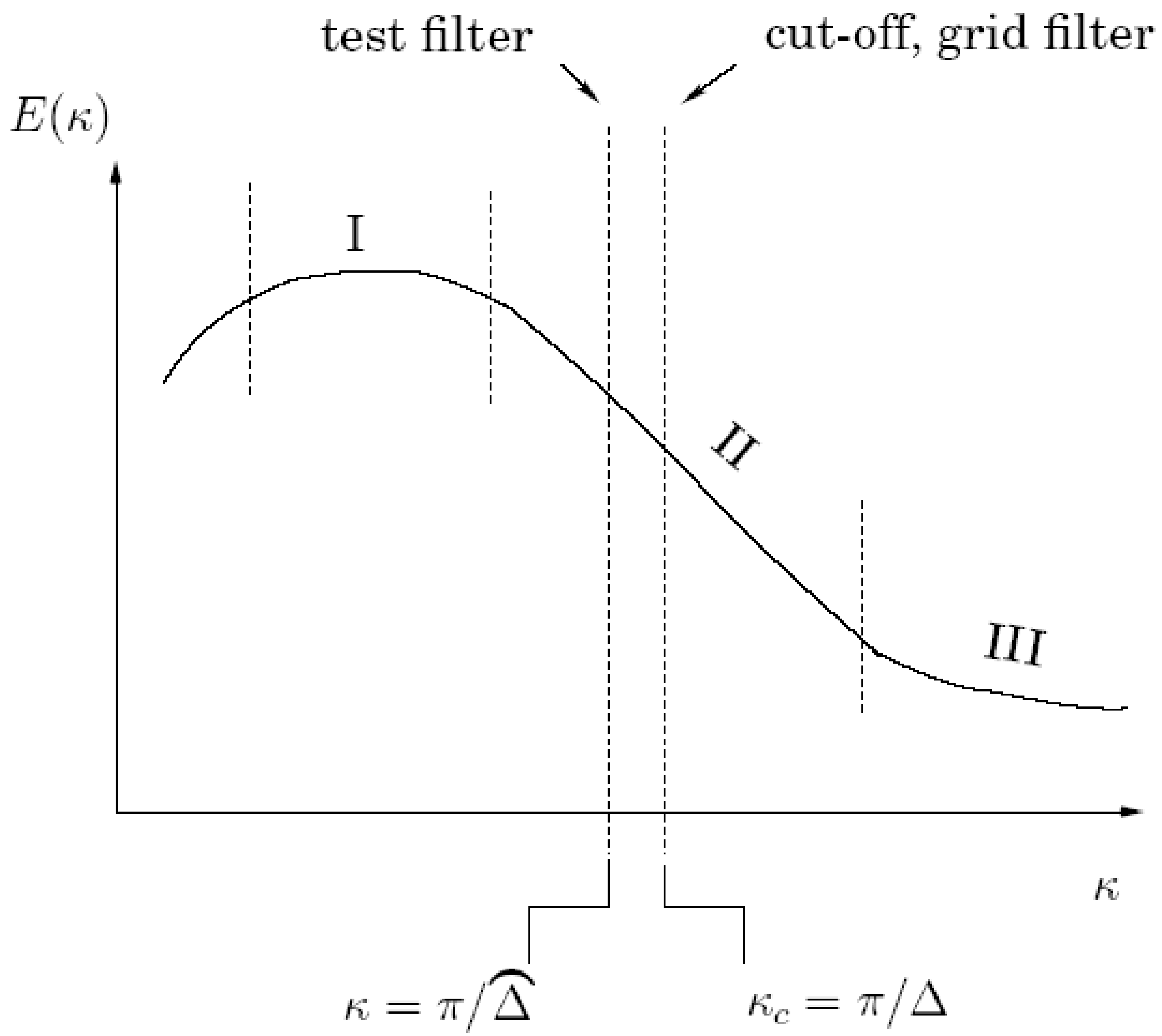
After second filter

$$\frac{\partial \widehat{\bar{u}}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\widehat{\bar{u}}_i \widehat{\bar{u}}_j\right) = -\frac{1}{\rho}\frac{\partial \widehat{\bar{p}}}{\partial x_i} + \nu \frac{\partial^2 \widehat{\bar{u}}_i}{\partial x_j \partial x_j} - \frac{\partial \widehat{\tau}_{ij}}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\widehat{\bar{u}_i \bar{u}_j} - \widehat{\bar{u}}_i \widehat{\bar{u}}_j\right)$$

$$\widehat{\bar{u}_i \bar{u}_j} - \widehat{\bar{u}}_i \widehat{\bar{u}}_j + \widehat{\tau}_{ij} = T_{ij}$$

The *dynamic* Leonard stresses are now defined as

$$\mathcal{L}_{ij} \equiv \widehat{\bar{u}_i \bar{u}_j} - \widehat{\bar{u}}_i \widehat{\bar{u}}_j = T_{ij} - \widehat{\tau}_{ij}$$



In the energy spectrum, the test filter is located at lower wave number than the grid filter, see the figure above.

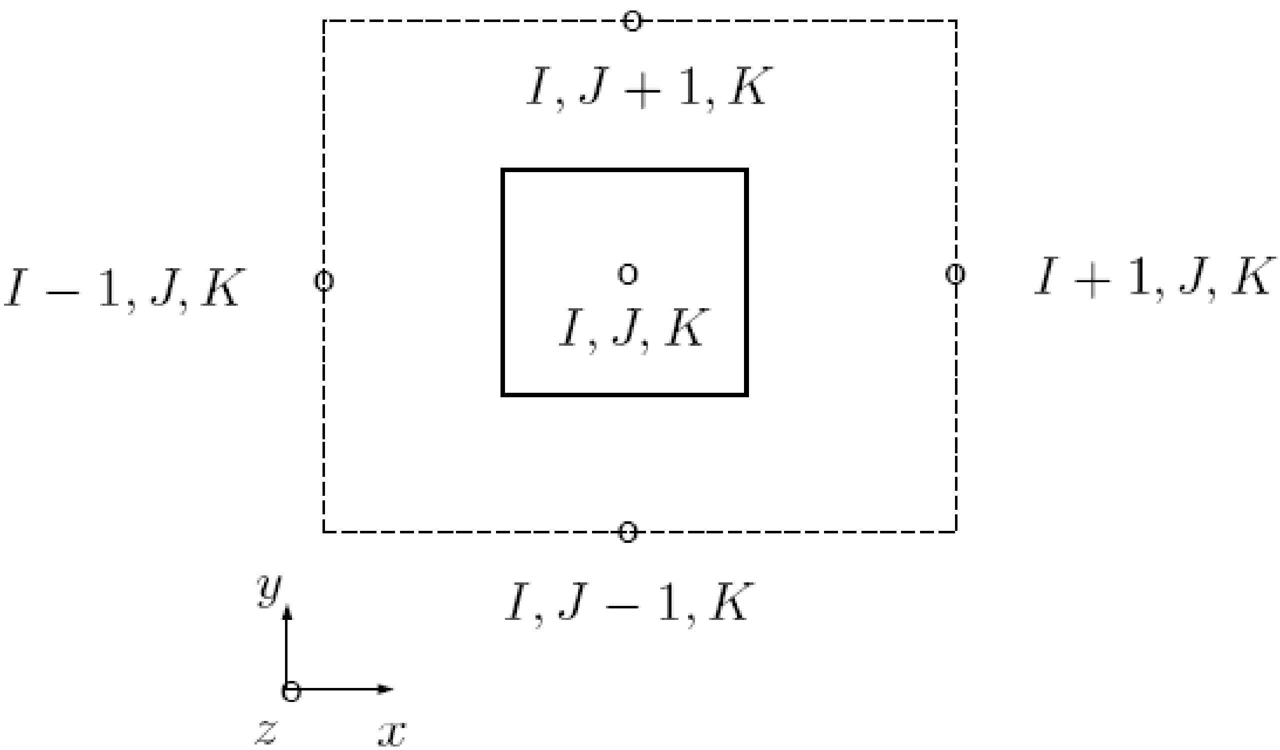
The test filter is twice the size of the grid filter, i.e. $\widehat{\Delta} = 2\Delta$.

The test-filtered variables are computed by integration over the test filter. For example, in the 1D example above $\widehat{\bar{u}}$ is computed as ($\widehat{\Delta x} = 2\Delta x$)

$$\begin{aligned}\widehat{\bar{u}} &= \frac{1}{2\Delta x}\int_W^E \bar{u}dx = \frac{1}{2\Delta x}\left(\int_W^P \bar{u}dx + \int_P^E \bar{u}dx\right)\\ &= \frac{1}{2\Delta x}\left(\bar{u}_w\Delta x + \bar{u}_e\Delta x\right) = \frac{1}{2}\left(\frac{\bar{u}_W + \bar{u}_P}{2} + \frac{\bar{u}_P + \bar{u}_E}{2}\right)\\ &= \frac{1}{4}\left(\bar{u}_W + 2\bar{u}_P + \bar{u}_E\right)\end{aligned}$$

For 3D, filtering at the test level is carried out in the same way by integrating over the test cell assuming linear variation of the variables (Zang *et al.*, 1993), i.e.

$$\widehat{\bar{u}}_{I,J,K} = \frac{1}{8}(\bar{u}_{I-1/2,J-1/2,K-1/2} + \bar{u}_{I+1/2,J-1/2,K-1/2}$$
$$+\bar{u}_{I-1/2,J+1/2,K-1/2} + \bar{u}_{I+1/2,J+1/2,K-1/2}$$
$$+\bar{u}_{I-1/2,J-1/2,K+1/2} + \bar{u}_{I+1/2,J-1/2,K+1/2}$$
$$+\bar{u}_{I-1/2,J+1/2,K+1/2} + \bar{u}_{I+1/2,J+1/2,K+1/2})$$



Stresses on grid, test and intermediate level

The stresses on the grid level, test level and intermediate level (dynamic Leonard stresses) have the form

$$\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j \quad \text{stresses with } \ell < \Delta$$

$$T_{ij} = \widehat{\overline{u_i u_j}} - \widehat{\bar{u}}_i \widehat{\bar{u}}_j \quad \text{stresses with } \ell < \widehat{\Delta}$$

$$\mathcal{L}_{ij} = T_{ij} - \widehat{\tau}_{ij} \quad \text{stresses with } \Delta < \ell < \widehat{\Delta}$$

Thus the dynamic Leonard stresses represent the contribution to the stresses with scales ℓ in the range between Δ and $\widehat{\Delta}$.

If we use the Smagorinsky model we get

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -2C\Delta^2|\bar{s}|\bar{s}_{ij}$$

$$T_{ij} - \frac{1}{3}\delta_{ij}T_{kk} = -2C\widehat{\Delta}^2|\widehat{\bar{s}}|\widehat{\bar{s}}_{ij}$$

$$\widehat{\bar{s}}_{ij} = \frac{1}{2}\left(\frac{\partial \widehat{\bar{u}}_i}{\partial x_j} + \frac{\partial \widehat{\bar{u}}_j}{\partial x_i}\right) \qquad |\widehat{\bar{s}}| = \left(2\widehat{\bar{s}}_{ij}\widehat{\bar{s}}_{ij}\right)^{1/2}$$

$$\mathcal{L}_{ij} \equiv \widehat{\bar{u}_i \bar{u}_j} - \widehat{\bar{u}}_i \widehat{\bar{u}}_j = T_{ij} - \widehat{\tau}_{ij}$$

$$T_{ij} - \frac{1}{3}\delta_{ij} T_{kk} = -2C\widehat{\Delta}^2 |\widehat{\bar{s}}| \widehat{\bar{s}}_{ij}$$

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -2C\Delta^2 |\bar{s}|\bar{s}_{ij}$$

assuming that C varies slowly

$$\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} = -2C\left(\widehat{\Delta}^2 |\widehat{\bar{s}}| \widehat{\bar{s}}_{ij} - \Delta^2 \widehat{|\bar{s}|\bar{s}_{ij}}\right)$$

Note that the 'constant' C really is a function of both space and time, i.e. $C = C(x_i, t)$.

This equation is a tensor equation, and we have five ($\bar{s}_{ij}$ is symmetric and trace-less) equations for C. Lilly (1992) suggested to satisfy this equation n a least-square sense, defining the error Q as the difference between the left-hand side and the right-hand side of this equation raised to the power of two, i.e.

$$Q = \left(\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} + 2CM_{ij}\right)^2 \qquad M_{ij} = \left(\widehat{\Delta}^2 |\widehat{\bar{s}}| \widehat{\bar{s}}_{ij} - \Delta^2 \widehat{|\bar{s}|\bar{s}_{ij}}\right)$$

and requiring $\partial Q/\partial C = 0$, which gives

$$C = -\frac{\mathcal{L}_{ij}M_{ij}}{2M_{ij}M_{ij}}$$

It turns out the the dynamic coefcient C fluctuates wildly both in space and time. This causes numerical problems,and it has been found necessary to average C in homogeneous direction(s). Furthermore, C must be clipped to ensure that the total viscosity stays positive $(\nu + \nu_{sgs} \geq 0)$.

In real 3D flows, there is no homogeneous direction. This makes it very difficult (read: impossible) to use this model, without introducing some arbitrary averaging and clipping.

Scale-similarity Models

In the models presented in the previous sections (the Smagorinsky and the dynamic models) the total SGS stress $\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j$ was modelled with an eddy-viscosity hypothesis. In scale-similarity models the total stress is split up as

$$\begin{aligned}
\tau_{ij} &= \overline{u_i u_j} - \bar{u}_i \bar{u}_j = \overline{(\bar{u}_i + u_i'')(\bar{u}_j + u_j'')} - \bar{u}_i \bar{u}_j \\
&= \overline{\bar{u}_i \bar{u}_j} + \overline{\bar{u}_i u_j''} + \overline{\bar{u}_j u_i''} + \overline{u_i'' u_j''} - \bar{u}_i \bar{u}_j \\
&= (\overline{\bar{u}_i \bar{u}_j} - \bar{u}_i \bar{u}_j) + \left[\overline{\bar{u}_i u_j''} + \overline{\bar{u}_j u_i''}\right] + \overline{u_i'' u_j''}
\end{aligned}$$

$$\tau_{ij} = L_{ij} + C_{ij} + R_{ij}$$

$$L_{ij} = \overline{\bar{u}_i \bar{u}_j} - \bar{u}_i \bar{u}_j$$

$$C_{ij} = \overline{\bar{u}_i u_j''} + \overline{\bar{u}_j u_i''}$$

$$R_{ij} = \overline{u_i'' u_j''}.$$

Note that the Leonard stresses L_{ij} are *computable*, i.e. they are exact and don't need to be modelled.

In scale-similarity models the main idea is that the turbulent scales just below (smaller than) Δ are similar to the ones just above Δ (hence the word "scale-similar"). Looking at Slide 24 it seems natural to assume that the cross term is responsible for the interaction between resolved scales ($\bar{u}_i$) and modelled scales (u_i''), since C_{ij} includes both scales.

The Bardina Model

Leonard stresses L_{ij} are computed explicitly.

$C_{ij}^M = c_r(\bar{u}_i\bar{u}_j - \overline{\bar{u}_i\bar{u}_j})$ and $R_{ij} = 0$ (superscript M denotes Modelled)

this model was not sufficiently dissipative

Smagorinsky model was added

$$C_{ij}^M = c_r(\bar{u}_i\bar{u}_j - \overline{\bar{u}_i\bar{u}_j})$$
$$R_{ij}^M = -2C_S\Delta^2|\bar{s}|\bar{s}_{ij}$$

Galilean invariance

Speziale (1985) found that the Leonard term L_{ij} and the cross term C_{ij} are not Galilean invariant by themselves, but only the sum $L_{ij} + C_{ij}$ is. As a consequence, if the cross term is neglected, the Leonard stresses must not be computed explicitly, because then the modelled momentum equations do not satisfy Galilean invariance.

Galilean invariance means that

the equations do not change if the coordinate system is moving with a constant speed V_k. Let's denote the moving coordinate system by $*$, i.e.

$$x_k^* = x_k + V_k t,\ t^* = t,\ \bar{u}_k^* = \bar{u}_k + V_k$$

By differentiating a variable ϕ we get

$$\frac{\partial \phi}{\partial x_k} = \frac{\partial x_j^*}{\partial x_k}\frac{\partial \phi}{\partial x_j^*} + \frac{\partial t^*}{\partial x_k}\frac{\partial \phi}{\partial t^*} = \frac{\partial \phi}{\partial x_k^*}$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial x_k^*}{\partial t}\frac{\partial \phi}{\partial x_k^*} + \frac{\partial t^*}{\partial t}\frac{\partial \phi}{\partial t^*} = V_k\frac{\partial \phi}{\partial x_k^*} + \frac{\partial \phi}{\partial t^*}.$$

easy to show that the Navier-Stokes (both with and without filter) is Galilean invariant (Speziale,1985; Panton, 1984).

$$\begin{aligned}\frac{\partial \phi}{\partial t} + u_k\frac{\partial \phi}{\partial x_k} &= \frac{\partial \phi}{\partial t^*} + V_k\frac{\partial \phi}{\partial x_k^*} + (u_k^* - V_k)\frac{\partial \phi}{\partial x_k^*} \\ &= \frac{\partial \phi}{\partial t^*} + u_k^*\frac{\partial \phi}{\partial x_k^*},\end{aligned}$$

Now, let's look at the Leonard term and the cross term.

Since the filtering operation is Galilean invariant (Speziale, 1985), we have $\bar{u}_k^* = \bar{u}_k + V_k$ and consequently also $u_k''^* = u_k''$.

V_i is constant $V_i = \bar{V}_i = \bar{\bar{V}}_i$)

$$
\begin{aligned}
L_{ij}^* &= \overline{\bar{u}_i^* \bar{u}_j^*} - \bar{u}_i^* \bar{u}_j^* = \overline{(\bar{u}_i + V_i)(\bar{u}_j + V_j)} - (\bar{u}_i + V_i)(\bar{u}_j + V_j) \\
&= \overline{\bar{u}_i \bar{u}_j} + \overline{\bar{u}_i} V_j + \overline{\bar{u}_j} V_i - \bar{u}_i \bar{u}_j - \bar{u}_i V_j - V_i \bar{u}_j \\
&= \overline{\bar{u}_i \bar{u}_j} - \bar{u}_i \bar{u}_j + V_j (\overline{\bar{u}_i} - \bar{u}_i) + V_i (\overline{\bar{u}_j} - \bar{u}_j) \\
&= L_{ij} - V_j \overline{u_i''} - V_i \overline{u_j''}
\end{aligned}
$$

$$
\begin{aligned}
C_{ij}^* &= \overline{\bar{u}_i^* u_j''^*} + \overline{\bar{u}_j^* u_i''^*} = \overline{(\bar{u}_i + V_i) u_j''} + \overline{(\bar{u}_j + V_j) u_i''} = \\
&= \overline{\bar{u}_i u_j''} + \overline{u_j''} V_i + \overline{\bar{u}_j u_i''} + \overline{u_i''} V_j = C_{ij} + \overline{u_j''} V_i + \overline{u_i''} V_j
\end{aligned}
$$

are not Galilean invariant

$$L^*_{ij} + C^*_{ij} = L_{ij} + C_{ij}.$$

The requirement for the Bardina model to be Galilean invariant is that the constant must be one, $c_r = 1$

$$\begin{aligned}
C^{*M}_{ij} &= c_r(\bar{u}^*_i \bar{u}^*_j - \overline{\bar{u}}^*_i \overline{\bar{u}}^*_j) \\
&= c_r\left[(\bar{u}_i + V_i)(\bar{u}_j + V_j) - \overline{(\bar{u}_i + V_i)}\,\overline{(\bar{u}_j + V_j)}\right] \\
&= c_r\left[\bar{u}_i\bar{u}_j - \overline{\bar{u}_i}\overline{\bar{u}_j} - (\overline{\bar{u}}_i - \bar{u}_i)V_j - (\overline{\bar{u}}_j - \bar{u}_j)V_i\right] \\
&= C^M_{ij} + c_r\left[\overline{u''}_i V_j + \overline{u''}_j V_i\right].
\end{aligned}$$

As is seen, $C^{*M}_{ij} \neq C^M_{ij}$

$c_r = 1 \;\Rightarrow\; C^{*M}_{ij} + L^*_{ij} = C^M_{ij} + L_{ij}.$

Note that in order to make the Bardina model Galilean invariant the Leonard stress *must* be computed explicitly.

Redefined terms in the Bardina Model

The stresses in the Bardina model can be redefined to make it Galilean invariant for any value c_r. A modified Leonard stress tensor L_{ij}^m is defined as (Germano, 1986)

$$\tau_{ij}^m = \tau_{ij} = C_{ij}^m + L_{ij}^m + R_{ij}^m$$

$$L_{ij}^m = c_r \left(\overline{\bar{u}_i \bar{u}_j} - \bar{\bar{u}}_i \bar{\bar{u}}_j \right)$$

$$L_{ij}^m = L_{ij} + C_{ij}^M$$

$$C_{ij}^m = 0$$

$$R_{ij}^m = R_{ij} = \overline{u_i'' u_j''}$$

In order to make the model sufficiently dissipative a Smagorinsky model is added, and the total SGS stress τ_{ij} is modelled as

$$\tau_{ij} = \overline{\bar{u}_i \bar{u}_j} - \bar{\bar{u}}_i \bar{\bar{u}}_j - 2C\Delta |\bar{s}| \bar{s}_{ij}$$

$$\frac{1}{c_r}L_{ij}^{m*} = \overline{\bar{u}_i^*\bar{u}_j^*} - \bar{\bar{u}}_i^*\bar{\bar{u}}_j^* = \overline{(\bar{u}_i + V_i)(\bar{u}_j + V_j)} - \overline{(\bar{u}_i + V_i)}\;\overline{(\bar{u}_j + V_j)}$$

$$= \overline{\bar{u}_i\bar{u}_j} + \bar{\bar{u}}_iV_j + \bar{\bar{u}}_jV_i - \bar{\bar{u}}_i\bar{\bar{u}}_j - \bar{\bar{u}}_iV_j - V_i\bar{\bar{u}}_j$$

$$= \overline{\bar{u}_i\bar{u}_j} - \bar{\bar{u}}_i\bar{\bar{u}}_j = \frac{1}{c_r}L_{ij}^m$$

we verify that the modied Leonard stress is Galilean invariant.

One-equation k_{sgs} model

A one-equation model can be used to model the SGS turbulent kinetic energy. The equation can be written on the same form as the RANS k-equation, i.e.

$$\frac{\partial k_{sgs}}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_j k_{sgs}) = \frac{\partial}{\partial x_j}\left[(\nu + \nu_{sgs})\frac{\partial k_{sgs}}{\partial x_j}\right] + P_{k_{sgs}} - C_\varepsilon \frac{k_{sgs}^{3/2}}{\Delta}$$

$$\nu_{sgs} = c_k \Delta k_{sgs}^{1/2}, \;\; P_{k_{sgs}} = 2\nu_{sgs}\bar{s}_{ij}\bar{s}_{ij}$$

Note that the production term, $P_{k_{sgs}}$, is equivalent to the SGS dissipation in the equation for the resolved turbulent kinetic energy

$$k_{sgs} = \frac{3}{2}C\left(\frac{\Delta\varepsilon}{\pi}\right)^{2/3}$$

$$\varepsilon = C_\varepsilon k_{sgs}^{3/2}/\Delta$$

$$k_{sgs} = \frac{3}{2}C\left(\frac{C_\varepsilon k_{sgs}^{3/2}}{\pi}\right)^{2/3}$$

so that $C_\varepsilon = \pi(\frac{3}{2}C)^{-3/2}$ which with $C = 1.5$ gives $C_\varepsilon = 0.93$.

$$\nu_{sgs} = c_k \Delta k_{sgs}^{1/2}$$

$$2\nu_{sgs}\bar{s}_{ij}\bar{s}_{ij} = \varepsilon$$

$$\nu_{sgs} = (C_S\Delta)^2|\bar{s}|$$

$$c_k k_{sgs}^{1/2}\Delta|\bar{s}|^2 = \varepsilon$$

$$|\bar{s}|^2 = 2\int_0^{\kappa_c} \kappa^2 C\kappa^{-5/3}\varepsilon^{2/3}d\kappa = \frac{3}{2}\pi^{4/3}C\varepsilon^{2/3}\Delta^{-4/3}$$

$$k_{sgs} = \frac{3}{2}C\left(\frac{\Delta\varepsilon}{\pi}\right)^{2/3}$$

$$c_k\left(\frac{3}{2}C\left(\frac{\Delta\varepsilon}{\pi}\right)^{2/3}\right)^{1/2}\Delta\frac{3}{2}\pi^{4/3}C\varepsilon^{2/3}\Delta^{-4/3} = \varepsilon \;\Rightarrow\; c_k = \left(\tfrac{3}{2}C\right)^{-3/2}/\pi = 0.094$$

$$C = 1.5$$

A dynamic one-equation model

One of the drawbacks of the dynamic model of Germano *et al.* (1991) (see p. 59) is the numerical instability associated with the negative values and large variation of the C coefficient. Usually this problem is fixed by averaging the coefficient in some homogeneous flow direction. However, in real applications, no such flow direction exists. Below a dynamic one-equation model is presented. The main object when developing this model was that it should be applicable to real industrial flows. Furthermore, being a dynamic model, it has the great advantage that the coefficients are *computed* rather than being prescribed.

$$\frac{\partial k_{sgs}}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_j k_{sgs}) = P_{k_{sgs}} + \frac{\partial}{\partial x_j}\left(\nu_{eff}\frac{\partial k_{sgs}}{\partial x_j}\right) - C_* \frac{k_{sgs}^{3/2}}{\Delta}$$

$$P_{k_{sgs}} = -\tau_{ij}^a \bar{u}_{i,j}, \quad \tau_{ij}^a = -2C\Delta k_{sgs}^{\frac{1}{2}} \bar{s}_{ij} \qquad \nu_{eff} = \nu + 2C_{hom}\Delta k_{sgs}^{\frac{1}{2}}$$

The C in the production term P_{ksgs} is computed dynamically $C = -\frac{\mathcal{L}_{ij} M_{ij}}{2 M_{ij} M_{ij}}$

To ensure numerical stability, a *constant* value (in space) of C (C_{hom}) is used in the diffusion term and in the momentum equations. C_{hom} is computed by requiring that C_{hom} should yield the same total production of k_{sgs} as C, i.e.

$$\langle 2C\Delta k_{sgs}^{\frac{1}{2}} \bar{s}_{ij} \bar{s}_{ij} \rangle_{xyz} = 2C_{hom} \langle \Delta k_{sgs}^{\frac{1}{2}} \bar{s}_{ij} \bar{s}_{ij} \rangle_{xyz}$$

The dissipation term $\varepsilon_{k_{sgs}}$ is estimated as:

$$\varepsilon_{k_{sgs}} \equiv \nu \mathcal{T}_f(u_{i,j}, u_{i,j}) = C_* \frac{k_{sgs}^{3/2}}{\Delta}.$$

Now we want to find a dynamic equation for C_*. The equations for k_{sgs} and K read in symbolic form

$$T(k_{sgs}) \equiv C_{k_{sgs}} - D_{k_{sgs}} = P_{k_{sgs}} - C_* \frac{k_{sgs}^{3/2}}{\Delta}$$

$$T(K) \equiv C_K - D_K = P_K - C_* \frac{K^{3/2}}{\widehat{\Delta}}$$

Since the turbulence on both the grid level and the test level should be in local equilibrium (in the inertial $-5/3$ region), the left-hand side of the two equations

should be close to zero. An even better approximation should be to assume $T(k_{sgs}) = T(K)$, i.e.

$$\widehat{P}_{k_{sgs}} - \frac{1}{\Delta}\widehat{C_* k_{sgs}}^{3/2} = P_K - C_* \frac{K^{3/2}}{\widehat{\Delta}},$$

so that

$$C_*^{n+1} = \left(P_K - \widehat{P}_{k_{sgs}} + \frac{1}{\Delta}\widehat{C_*^n k_{sgs}^{3/2}} \right) \frac{\widehat{\Delta}}{K^{\frac{3}{2}}}.$$

A Mixed Model Based on a One-Eq. Model

Recently a new dynamic scale-similarity model was presented by Krajnovi´c & Davidson (2002*b*). In this model a dynamic one-equation SGS model is solved, and the scalesimilarity part is estimated.