بسم الله الرحمن الرحیم

Prediction of Turbulent flow – Part 2

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Probability Density Function

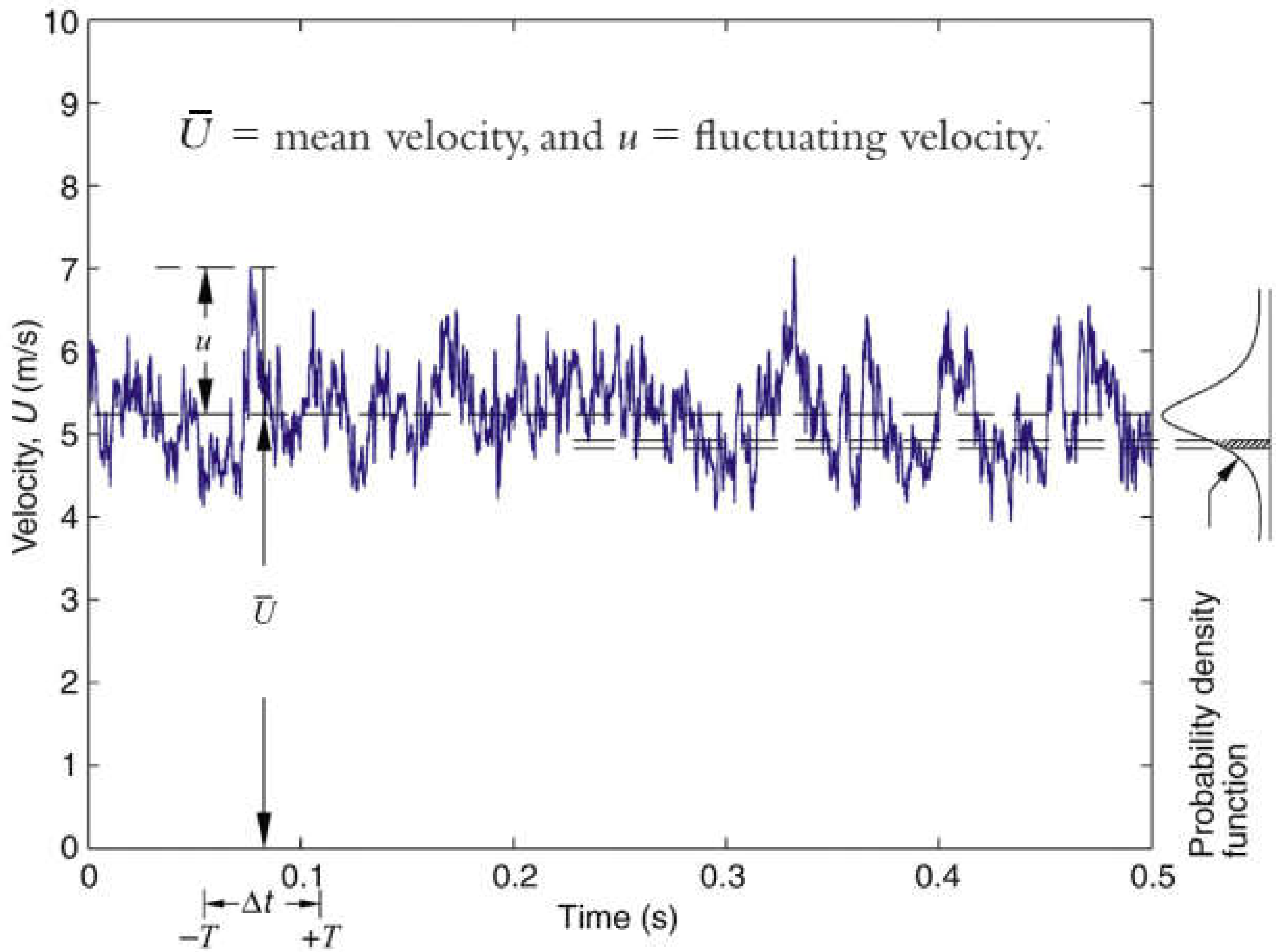


Figure 3.1 A "stationary" random flow event. *(Created by N. Cao).*

The probability of a random event, Event A for example, can be written as

$$p = P(A) = P\{4.8\,\mathrm{m/s} \le U < 4.9\,\mathrm{m/s}\}$$

where p is a real number between 0 and 1, that is, $0 \le p \le 1$

The *root-mean-square* (RMS) can be defined from the second moment as

$$v_{rms} = \left(\overline{v'^2}\right)^{1/2}$$

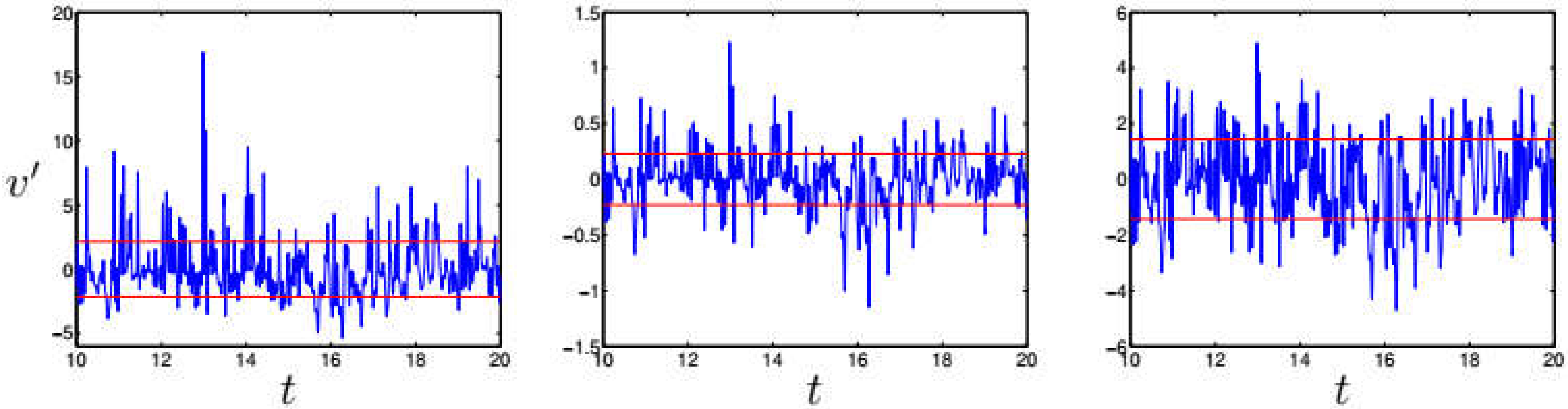
The RMS is the same as the *standard deviation* which is equal to the square-root of the *variance*.

probability densities (sometimes called *histograms*)

$$\bar{v} = \int_{-\infty}^{\infty} v f_v(v) dv \qquad \int_{-\infty}^{\infty} f_v(v) dv = 1$$

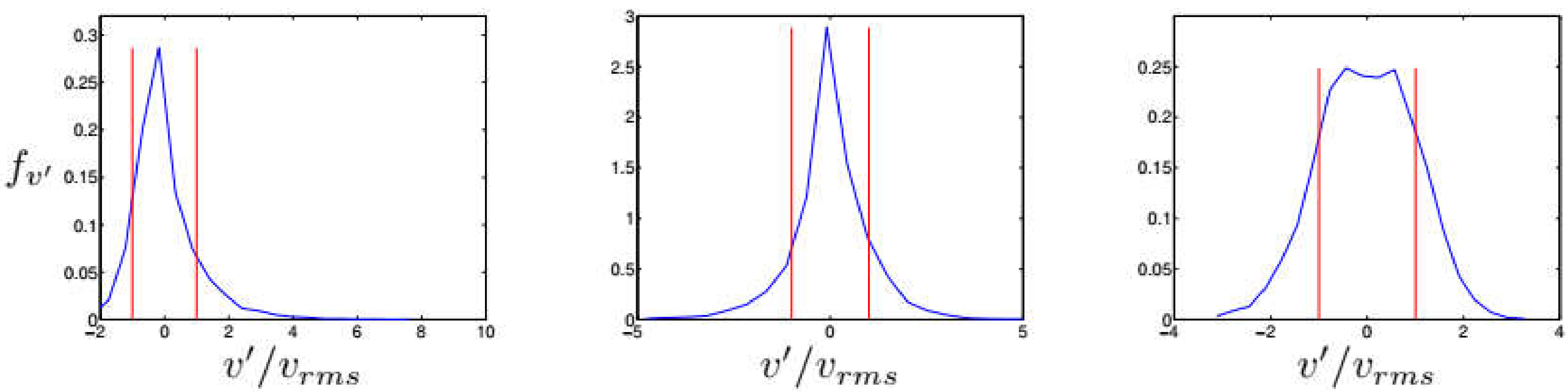
the variance of the fluctuations

$$\overline{v'^2} = \int_{-\infty}^{\infty} v'^2 f_{v'}(v') dv'$$



(a) Point 1. $v_{rms} = 2.15$.   (b) Point 2. $v_{rms} = 0.23$.   (c) Point 3. $v_{rms} = 1.44$.

Figure 7.1: Time history of v'. Horizontal red lines show $\pm v_{rms}$.



(a) Point 1. $S = 1.51$ and $F = 7.68$.   (b) Point 2. $S = -0.27$ and $F = 5.81$.   (c) Point 3. $S = -0.09$ and $F = 2.77$.

Figure 7.2: Probability density functions of time histories in Fig. 7.1. Vertical red lines show $\pm v_{rms}$. The skewness, S, and the flatness, F, are given for the three time histories.

Point 1. The time history of the velocity fluctuation (Fig. 7.1a) shows that there exists large positive values but no large negative values. The positive values are often larger than $+v_{rms}$ (the peak is actually close to $8v_{rms}$) but the negative values are seldom smaller than $-v_{rms}$. This indicates that the distribution of v' is skewed towards the positive side. This is confirmed in the PDF distribution, see Fig. 7.2a.

Point 2. The fluctuations at this point are much smaller and the positive values are as large the negative values; this means that the PDF should be symmetric which is confirmed in Fig. 7.2b. The extreme values of v' are approximately $\pm 1.5v_{rms}$, see Figs. 7.1b and 7.2b.

Point 3. At this point the time history (Fig. 7.1c) shows that the fluctuations are clustered around zero and much values are within $\pm v_{rms}$. The time history shows that the positive and the negative values have the same magnitude. The PDF function in Fig. 7.2c confirms that there are many value around zero, that the extreme value are small and that positive and negative values are equally frequent (i.e. the PDF is symmetric).

In Fig. 7.2 we can judge whether the PDF is symmetric, but instead of "looking" at the probability density functions, we should use a definition of the degree of symmetry, which is the *skewness*. It is defined as

$$\overline{v'^3} = \int^{\infty} v'^3 f_{v'}(v')dv'$$

and is commonly normalized by v^3_{rms}, so that the skewness, $S_{v'}$, of v' is defined as

$$S_{v'} = \frac{1}{v^3_{rms}} \int_{-\infty}^{\infty} v'^3 f_{v'}(v')dv' = \frac{1}{2v^3_{rms}T} \int_{-T}^{T} v'^3(t)dt$$

More details can be found as:
Basic of Engineering Turbulence, 2000, pp. 50- 64

For a Gaussian distribution

$$f(v') = \frac{1}{v_{rms}} \exp\left(-\frac{(v' - v_{rms})^2}{2v^2_{rms}}\right)$$

namely the *flatness*.

$$F = \frac{1}{v^4_{rms}} \int_{-\infty}^{\infty} v'^4 f_{v'}(v)dv$$

The Exact k Equation

The equation for turbulent kinetic energy, $k = \frac{1}{2}\overline{v_i' v_i'}$, is derived from the Navier-Stokes

$$\rho \frac{\partial v_i}{\partial t} + \rho \frac{\partial v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} \quad - \quad \rho \frac{\partial \bar{v}_i \bar{v}_j}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \bar{v}_i}{\partial x_j} - \rho \overline{v_i' v_j'}\right)$$

حاصل فوق را در v_i' ضرب و بر دانسیته تقسیم نموده و در انتها متوسط زمانی گرفته شود نتیجه خواهد شد

$$\overline{v_i' \frac{\partial}{\partial x_j}\left[v_i v_j - \bar{v}_i \bar{v}_j\right]} = -\frac{1}{\rho}\overline{v_i' \frac{\partial}{\partial x_i}\left[p - \bar{p}\right]} + \nu \overline{v_i' \frac{\partial^2}{\partial x_j \partial x_j}\left[v_i - \bar{v}_i\right]} + \overline{\frac{\partial \overline{v_i' v_j'}}{\partial x_j} v_i'}$$

$$\frac{1}{2}\frac{\partial}{\partial x_j}\overline{(v_j' v_i' v_i')}.$$

Using $v_j = \bar{v}_j + v_j'$

$$\overline{v_i' \frac{\partial}{\partial x_j}\left[(\bar{v}_i + v_i')(\bar{v}_j + v_j') - \bar{v}_i \bar{v}_j\right]} = \overline{v_i' \frac{\partial}{\partial x_j}\left[\bar{v}_i v_j' + v_i' \bar{v}_j + v_i' v_j'\right]}.$$

$\partial v_j' / \partial x_j = 0$

$$\overline{v_i' \frac{\partial}{\partial x_j}(v_i' \bar{v}_j)} = \bar{v}_j \overline{v_i' \frac{\partial v_i'}{\partial x_j}}$$

$\partial \bar{v}_j / \partial x_j = 0$

$$\overline{v_i' \frac{\partial}{\partial x_j}(\bar{v}_i v_j')} = \overline{v_i' v_j'} \frac{\partial \bar{v}_i}{\partial x_j}$$

$$\bar{v}_j \left(\overline{v_i' \frac{\partial v_i'}{\partial x_j}}\right) = \bar{v}_j \frac{\partial}{\partial x_j}\overline{\left(\frac{1}{2} v_i' v_i'\right)} = \bar{v}_j \frac{\partial}{\partial x_j}(k) = \frac{\partial}{\partial x_j}(\bar{v}_j k)$$

$$\overline{v_i'\frac{\partial}{\partial x_j}\left[(\bar{v}_i+v_i')(\bar{v}_j+v_j')-\bar{v}_i\bar{v}_j\right]} = \overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\bar{v}_j k) + \frac{1}{2}\frac{\partial}{\partial x_j}\overline{(v_j'v_i'v_i')}$$

$$\boxed{-\frac{1}{\rho}\overline{v_i'\frac{\partial}{\partial x_i}[p-\bar{p}]}} + \boxed{\nu\overline{v_i'\frac{\partial^2}{\partial x_j\partial x_j}[v_i-\bar{v}_i]}} + \overset{=0}{\cancel{\overline{\frac{\partial \overline{v_i'v_j'}}{\partial x_j}v_i'}}}$$

سمت راست معادله بدست آمده در اسلاید صفحه قبل

$$\overline{\bar{a}b'} = \bar{a}\bar{b'} = 0.$$

$$-\frac{1}{\rho}\overline{v_i'\frac{\partial p'}{\partial x_i}} = -\frac{1}{\rho}\frac{\partial\overline{p'v_i'}}{\partial x_i} + \overset{=0}{\cancel{\frac{1}{\rho}\overline{p'\frac{\partial v_i'}{\partial x_i}}}} = -\frac{1}{\rho}\frac{\partial\overline{p'v_i'}}{\partial x_i}$$

$$\nu\overline{v_i'\frac{\partial^2 v_i'}{\partial x_j\partial x_j}} = \nu\overline{v_i'\frac{\partial}{\partial x_2}\left(\frac{\partial v_i'}{\partial x_j}\right)} = \nu\overline{\frac{\partial}{\partial x_j}\left(v_i'\frac{\partial v_i'}{\partial x_j}\right)} - \nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$

$(A = v_i'$ and $B = \partial v_i'/\partial x_j)$

$$\nu\overline{\frac{\partial}{\partial x_j}\left(v_i'\frac{\partial v_i'}{\partial x_j}\right)} = \nu\overline{\frac{\partial}{\partial x_j}\left(\frac{1}{2}\left(v_i'\frac{\partial v_i'}{\partial x_j}+v_i'\frac{\partial v_i'}{\partial x_j}\right)\right)} =$$

$$\nu\overline{\frac{\partial}{\partial x_j}\left(\frac{1}{2}\left(\frac{\partial v_i'v_i'}{\partial x_j}\right)\right)} = \nu\frac{1}{2}\frac{\partial^2\overline{v_i'v_i'}}{\partial x_j\partial x_j} = \nu\frac{\partial^2 k}{\partial x_j\partial x_j}$$

$$\underbrace{\frac{\partial \bar{v}_j k}{\partial x_j}}_{I} = \underbrace{-\overline{v_i' v_j'}\frac{\partial \bar{v}_i}{\partial x_j}}_{II} \underbrace{-\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j' p'} + \frac{1}{2}\overline{v_j' v_i' v_i'} - \nu \frac{\partial k}{\partial x_j}\right]}_{III} \underbrace{-\nu \overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}}_{IV}$$

I **Convection**.

II **Production**, P^k. The large turbulent scales extract energy from the mean flow. This term (including the minus sign) is almost always positive. It is largest for the energy-containing eddies, i.e. for small wavenumbers, see Fig. 5.2. This term originates from the convection term (the first term on the right side of Eq. 8.6). It can be noted that the production term is an acceleration term, $v_j'\partial \bar{v}_i/\partial x_j$, multiplied by a fluctuating velocity, v_i', i.e. the product of an inertial force per unit mass (acceleration) and a fluctuating velocity. A force multiplied with a velocity corresponds to work per unit time. When the acceleration term and the fluctuating velocity are in opposite directions (i.e. when $P^k > 0$), the mean flow performs work on the fluctuating velocity field. When the production term is **negative**, it means that the fluctuations are doing work on the mean flow field. In this case, v_j' and the acceleration term, $v_j'\partial \bar{v}_i/\partial x_j$, have the same sign.

$$\underbrace{\frac{\partial \bar{v}_j k}{\partial x_j}}_{I} = \underbrace{-\overline{v_i' v_j'}\frac{\partial \bar{v}_i}{\partial x_j}}_{II} \underbrace{- \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j' p'} + \frac{1}{2}\overline{v_j' v_i' v_i'} - \nu \frac{\partial k}{\partial x_j}\right]}_{III} \underbrace{- \nu \overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}}_{IV}$$

III The two first terms represent **turbulent diffusion** by pressure-velocity fluctuations, and velocity fluctuations, respectively. The last term is viscous diffusion. The velocity-fluctuation term originates from the convection term (the last term

IV **Dissipation**, ε. This term is responsible for transformation of kinetic energy at small scales to thermal energy. The term (excluding the minus sign) is always

positive (it consists of velocity gradients squared). It is largest for large wavenumbers, see Fig. 5.2. The dissipation term stems from the viscous term (see Eq. 8.12) in the Navier-Stokes equation. It can be written as $\overline{v_i' \partial \tau_{ij}' / \partial x_j}$, see Eq. 4.1. The divergence of τ_{ij}' is a force vector (per unit mass), i.e. $T_i' = \partial \tau_{ij}' / \partial x_j$. The dissipation term can now be written $\overline{v_i' T_i'}$, which is a scalar product between two vectors. When the viscous stress vector is in the opposite direction to the fluctuating velocity, the term is negative (i.e. it is dissipative); this means that the viscose stress vector performs work and transforms kinetic energy into internal energy.

$$\tau_{ij} = 2\mu S_{ij} - \frac{2}{3}\mu S_{kk}\delta_{ij}$$

$$\frac{\partial}{\partial x_j}(2\mu S_{ij}) = \mu \frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

$$\mu \frac{\partial}{\partial x_j}\left(\frac{\partial v_j}{\partial x_i}\right) = \mu \frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right) = 0.$$

$$\frac{\partial \tau_{ji}}{\partial x_j} = \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

$$\underbrace{\frac{\partial \bar{v}_j k}{\partial x_j}}_{I} = \underbrace{-\overline{v_i' v_j'}\frac{\partial \bar{v}_i}{\partial x_j}}_{II} \underbrace{-\frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\overline{v_j' p'} + \frac{1}{2}\overline{v_j' v_i' v_i'} - \nu\frac{\partial k}{\partial x_j}\right]}_{III} \underbrace{-\nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}}_{IV}$$

$$C^k = P^k + D^k - \varepsilon$$

where C^k, P^k, D^k and ε correspond to terms I-IV

The velocity gradient for an eddy characterized by velocity v_κ and lengthscale ℓ_κ can be estimated as

$$\left(\frac{\partial v}{\partial x}\right)_\kappa \propto \frac{v_\kappa}{\ell_\kappa} \propto \left(v_\kappa^2\right)^{1/2}\kappa$$

since $\ell_\kappa \propto \kappa^{-1}$

The dimension of wavenumber is one over length:

$$E \propto k_\kappa/\kappa \propto v_\kappa^2/\kappa \propto \kappa^{-5/3} \quad \Rightarrow \quad v_\kappa^2 \propto \kappa^{-2/3}$$

$$\boxed{\left(\frac{\partial v}{\partial x}\right)_\kappa \propto \left(\kappa^{-2/3}\right)^{1/2}\kappa \propto \kappa^{-1/3}\kappa \propto \kappa^{2/3}}$$

$$\varepsilon = \nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}} \quad \Rightarrow \quad \varepsilon(\kappa) \propto \left(\frac{\partial v}{\partial x}\right)_\kappa^2 \propto \kappa^{4/3},$$

i.e. $\varepsilon(\kappa)$ does indeed increase for increasing wavenumber.

The energy transferred from eddy-to-eddy per unit time in spectral space can also be used for estimating the velocity gradient of an eddy

The cascade process assumes that this energy transfer per unit time is the same for each eddy size

$$\varepsilon_\kappa = \varepsilon = v_\kappa^3/\ell_\kappa = \ell_\kappa^2/\tau_\kappa^3 = \ell_0^2/\tau_0^3$$

$$\ell_\kappa^2/\tau_\kappa^3 = \ell_0^2/\tau_0^3$$

$$\varepsilon_\kappa \sim \frac{v_\kappa^2}{t_\kappa} \sim \frac{v_\kappa^2}{\ell_\kappa/v_\kappa} \sim \frac{v_\kappa^3}{\ell_\kappa}$$

$$\ell_\kappa^2/\tau_\kappa^3 = \ell_0^2/\tau_0^3 \Longrightarrow \tau_\kappa = \left(\frac{\ell_\kappa}{\ell_0}\right)^{2/3} \tau_0$$

where τ_0 and ℓ_0 are constants (they are given by the flow we're looking at

$$\left(\frac{\partial v}{\partial x}\right)_\kappa \propto \frac{v_\kappa}{\ell_\kappa} \propto \tau_\kappa^{-1} \propto \ell_\kappa^{-2/3} \propto \kappa^{2/3},$$

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The exact $\overline{v_i' v_j'}$ equation

An exact equation for the Reynolds stresses can be derived from the Navies-Stokes equation

- Set up the momentum equation for the instantaneous velocity $v_i = \bar{v}_i + v_i' \rightarrow$ Eq. (A)
- Time average $\rightarrow$ equation for $\bar{v}_i$, Eq. (B)
- Subtract Eq. (B) from Eq. (A) $\rightarrow$ equation for v_i', Eq. (C)
- Do the same procedure for $v_j \rightarrow$ equation for v_j', Eq. (D)
- Multiply Eq. (C) with v_j' and Eq. (D) with v_i', time average and add them together $\rightarrow$ equation for $\overline{v_i' v_j'}$

The final $\overline{v'_i v'_j}$-equation (Reynolds Stress equation)

$$\frac{\partial \overline{v'_i v'_j}}{\partial t} + \underbrace{\bar{v}_k \frac{\partial \overline{v'_i v'_j}}{\partial x_k}}_{C_{ij}} = \underbrace{-\overline{v'_i v'_k} \frac{\partial \bar{v}_j}{\partial x_k} - \overline{v'_j v'_k} \frac{\partial \bar{v}_i}{\partial x_k}}_{P_{ij}} + \underbrace{\overline{\frac{p'}{\rho}\left(\frac{\partial v'_i}{\partial x_j} + \frac{\partial v'_j}{\partial x_i}\right)}}_{\Pi_{ij}}$$

$$\underbrace{-\frac{\partial}{\partial x_k}\left[\overline{v'_i v'_j v'_k} + \frac{\overline{p' v'_j}}{\rho}\delta_{ik} + \frac{\overline{p' v'_i}}{\rho}\delta_{jk}\right]}_{D_{ij,t}} + \underbrace{\nu \frac{\partial^2 \overline{v'_i v'_j}}{\partial x_k \partial x_k}}_{D_{ij,\nu}}$$

$$\underbrace{-g_i \beta \overline{v'_j \theta'} - g_j \beta \overline{v'_i \theta'}}_{G_{ij}} - \underbrace{2\nu \overline{\frac{\partial v'_i}{\partial x_k} \frac{\partial v'_j}{\partial x_k}}}_{\varepsilon_{ij}}$$

where $D_{ij,t}$ and $D_{ij,\nu}$ denote turbulent and viscous diffusion, respectively. The total diffusion reads $D_{ij} = D_{ij,t} + D_{ij,\nu}$. This is analogous to the momentum equation where we have gradients of viscous and turbulent stresses which correspond to viscous and turbulent diffusion.

$$C_{ij} = P_{ij} + \Pi_{ij} + D_{ij} + G_{ij} - \varepsilon_{ij}$$

C_{ij} Convection

P_{ij} Production

Π_{ij} Pressure-strain

D_{ij} Diffusion

G_{ij} Buoyancy production

ε_{ij} Dissipation

$$\underbrace{\frac{\partial \overline{v_i'v_j'}}{\partial t} + \bar{v}_k \frac{\partial \overline{v_i'v_j'}}{\partial x_k}}_{C_{ij}} = \underbrace{-\overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k}}_{P_{ij}} + \underbrace{\overline{\frac{p'}{\rho}\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)}}_{\Pi_{ij}}$$

$$\underbrace{-\frac{\partial}{\partial x_k}\left[\overline{v_i'v_j'v_k'} + \frac{\overline{p'v_j'}}{\rho}\delta_{ik} + \frac{\overline{p'v_i'}}{\rho}\delta_{jk}\right]}_{D_{ij,t}} + \underbrace{\nu\frac{\partial^2 \overline{v_i'v_j'}}{\partial x_k \partial x_k}}_{D_{ij,\nu}}$$

$$\underbrace{-g_i\beta\overline{v_j'\theta'} - g_j\beta\overline{v_i'\theta'}}_{G_{ij}} \underbrace{- 2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}}_{\varepsilon_{ij}} \qquad \text{Eq. } 11.11$$

Which terms in Eq. 11.11 are known and which are unknown? First, let's think about which physical quantities we solve for.

$\bar{v}_i$ is obtained from the momentum equation.

$\overline{v_i'v_j'}$ is obtained from the modeled $\overline{v_i'v_j'}$ equation.

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Hence the following terms in Eq. 11.11 are known (i.e. they do not need to be modeled)

- The left side
- The production term, P_{ij}
- The viscous part of the diffusion term, D_{ij}, i.e. D_{ij}^{ν}
- The buoyancy term, G_{ij} (provided that a transport equation is solved for $\overline{v_i'\theta'}$, Eq. 11.22; if not, $\overline{v_i'\theta'}$ is obtained from the Boussinesq assumption, Eq. 11.35)

$$\overline{v_i'\theta'} = -\alpha_t \frac{\partial \bar{\theta}}{\partial x_i}$$

$$\frac{\partial \overline{v_i'\theta'}}{\partial t} + \frac{\partial}{\partial x_k}\bar{v}_k \overline{v_i'\theta'} = \underbrace{-\overline{v_i'v_k'}\frac{\partial\bar{\theta}}{\partial x_k} - \overline{v_k'\theta'}\frac{\partial \bar{v}_i}{\partial x_k}}_{P_{i\theta}} \underbrace{-\frac{\overline{\theta'\,\partial p'}}{\rho\,\partial x_i}}_{\Pi_{i\theta}} \underbrace{-\frac{\partial}{\partial x_k}\overline{v_k'v_i'\theta'}}_{D_{i\theta,t}}$$

$$+ \underbrace{\left(\nu+\frac{\nu}{Pr}\right)\frac{\partial^2\overline{v_i'\theta'}}{\partial x_k\partial x_k}}_{D_{i\theta,\nu}} \underbrace{-(\nu+\alpha)\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial\theta'}{\partial x_k}}}_{\varepsilon_{i\theta}} \underbrace{-g_i\beta\overline{\theta'^2}}_{G_{i\theta}} \qquad (11.22)$$

$\bar{v}_i$ is obtained from the momentum equation,

$\bar{\theta}$ is obtained from the temperature equation, $$\frac{\partial\bar{\theta}}{\partial t} + \frac{\partial \bar{v}_i\bar{\theta}}{\partial x_i} = \alpha\frac{\partial^2\bar{\theta}}{\partial x_i\partial x_i} - \frac{\partial\overline{v_i'\theta'}}{\partial x_i}$$

$\overline{v_i'v_j'}$ is obtained from the modeled $\overline{v_i'v_j'}$ equation,

$\overline{v_i'\theta'}$ is obtained from the modeled $\overline{v_i'\theta'}$ equation

Hence the following terms in Eq. 11.22 are known (i.e. they do not need to be modeled)

- The left side
- The production term, $P_{i\theta}$
- The viscous diffusion term, $D_{i\theta,\nu}$
- The buoyancy term, $G_{i\theta}$ (provided that a transport equation is solved for $\overline{\theta'^2}$; if not, $\overline{\theta'^2}$ is usually modeled via a relation to k)

The k equation

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By taking the trace (setting indices $i=j$) of the equation for $\overline{v_i'v_j'}$ and dividing by two we get the equation for the turbulent kinetic energy:

$$\frac{\partial \overline{v_i'v_j'}}{\partial t} + \underbrace{\bar{v}_k \frac{\partial \overline{v_i'v_j'}}{\partial x_k}}_{C_{ij}} = \underbrace{-\overline{v_i'v_k'}\frac{\partial \bar{v}_j}{\partial x_k} - \overline{v_j'v_k'}\frac{\partial \bar{v}_i}{\partial x_k}}_{P_{ij}} + \underbrace{\overline{\frac{p'}{\rho}\left(\frac{\partial v_i'}{\partial x_j} + \frac{\partial v_j'}{\partial x_i}\right)}}_{\Pi_{ij}}$$

$$\underbrace{-\frac{\partial}{\partial x_k}\left[\overline{v_i'v_j'v_k'} + \frac{\overline{p'v_j'}}{\rho}\delta_{ik} + \frac{\overline{p'v_i'}}{\rho}\delta_{jk}\right]}_{D_{ij,t}} + \underbrace{\nu \frac{\partial^2 \overline{v_i'v_j'}}{\partial x_k \partial x_k}}_{D_{ij,\nu}}$$

$$\underbrace{-g_i\beta\overline{v_j'\theta'} - g_j\beta\overline{v_i'\theta'}}_{G_{ij}} - \underbrace{2\nu\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}}}_{\varepsilon_{ij}}$$

setting indices $i=j$

$$\frac{\partial k}{\partial t} + \underbrace{\bar{v}_j \frac{\partial k}{\partial x_j}}_{C^k} = \underbrace{-\overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j}}_{P^k} - \underbrace{\nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}}_{\varepsilon} \underbrace{-\frac{\partial}{\partial x_j}\left\{\overline{v_j'\left(\frac{p'}{\rho} + \frac{1}{2}v_i'v_i'\right)}\right\}}_{D_t^k} + \underbrace{\nu \frac{\partial^2 k}{\partial x_j \partial x_j}}_{D_\nu^k} \underbrace{-g_i\beta\overline{v_i'\theta'}}_{G^k}$$

$$C^k = P^k + D^k + G^k - \varepsilon \qquad D^k = D_t^k + D_\nu^k$$

$\bar{v}_i$ is obtained from the momentum equation,

k is obtained from the modeled k equation,

- The left side
- The viscous diffusion term, $D^k_{i,\nu}$
- The buoyancy term, G_{ij}

$$\overline{v_i'\theta'} = -\alpha_t \frac{\partial \bar{\theta}}{\partial x_i}$$

$$\underbrace{\frac{\partial k}{\partial t} + \bar{v}_j \frac{\partial k}{\partial x_j}}_{C^k} = \underbrace{-\overline{v_i' v_j'} \frac{\partial \bar{v}_i}{\partial x_j}}_{P^k} \underbrace{- \nu \overline{\frac{\partial v_i'}{\partial x_j} \frac{\partial v_i'}{\partial x_j}}}_{\varepsilon} \underbrace{- \frac{\partial}{\partial x_j}\left\{ \overline{v_j' \left(\frac{p'}{\rho} + \frac{1}{2} v_i' v_i' \right)} \right\}}_{D_t^k} + \underbrace{\nu \frac{\partial^2 k}{\partial x_j \partial x_j}}_{D_\nu^k} \underbrace{- g_i \beta \overline{v_i' \theta'}}_{G^k}$$

؟

یکی از سوالات در معادله بالا بحث اضمحلال (Dissipation) است. برای بدست آوردن معادله ε همانند معادله انرژی می توان عمل نمود.

Navier-Stokes equation

$$\rho \frac{\partial u_i}{\partial t} + \rho \frac{\partial}{\partial x_j}(u_j u_i) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(2\mu s_{ji})$$

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$\mathcal{N}(u_i) = \rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} - \mu \frac{\partial^2 u_i}{\partial x_k \partial x_k}$$

$$\mathcal{N}(u_i) = 0$$

The Modelled ε Equation

Navier-Stokes equation

دانشکده مهندسی مکانیک
دانشگاه صنعتی نوشیروانی بابل

$$\mathcal{N}(u_i) = \rho\frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} - \mu \frac{\partial^2 u_i}{\partial x_k \partial x_k} \qquad \mathcal{N}(u_i) = 0$$

$$\overline{2\nu \frac{\partial u_i'}{\partial x_j}\frac{\partial}{\partial x_j}\left[\mathcal{N}(u_i)\right]} = 0$$

Production of Dissipation

$$\rho\frac{\partial \epsilon}{\partial t} + \rho U_j \frac{\partial \epsilon}{\partial x_j} = -2\mu\left[\overline{u_{i,k}' u_{j,k}'} + \overline{u_{k,i}' u_{k,j}'}\right]\frac{\partial U_i}{\partial x_j} - 2\mu\, \overline{u_k' u_{i,j}'}\,\frac{\partial^2 U_i}{\partial x_k \partial x_j}$$

$$-2\mu\, \overline{u_{i,k}' u_{i,m}' u_{k,m}'} - 2\mu\,\nu\, \overline{u_{i,km}' u_{i,km}'}$$

Dissipation of Dissipation,

$$+\frac{\partial}{\partial x_j}\left[\mu\frac{\partial \epsilon}{\partial x_j} - \mu\, \overline{u_j' u_{i,m}' u_{i,m}'} - 2\nu\, \overline{p_{,m}' u_{j,m}'}\right]$$

Molecular Diffusion of Dissipation and Turbulent Transport of Dissipation,

This equation is far more complicated than the turbulence kinetic energy equation and involves several new unknown double and triple correlations of fluctuating velocity, pressure and velocity gradients.

I. Algebraic models.

II. One-equation models.

III. Two-equation models.

IV. Reynolds stress models

Non-linear Eddy-viscosity Models

$-\overline{u_i u_j}$

روشهای پیش بینی تنشهای رینولدز

I. **Algebraic models.** An algebraic equation is used to compute a turbulent viscosity, often called *eddy* viscosity. The Reynolds stress tensor is then computed using an assumption which relates the Reynolds stress tensor to the velocity gradients and the turbulent viscosity. This assumption is called the *Boussinesq assumption*. Models which are based on a turbulent (eddy) viscosity are called *eddy viscosity* mod-

II. **One-equation models.** In these models a transport equation is solved for a turbulent quantity (usually the turbulent kinetic energy) and a second turbulent quantity (usually a turbulent length scale) is obtained from an algebraic expression. The turbulent viscosity is calculated from Boussinesq assumption.

III. Two-equation models. These models fall into the class of eddy viscosity models. Two transport equations are derived which describe transport of two scalars, for example the turbulent kinetic energy k and its dissipation ε. The Reynolds stress tensor is then computed using an assumption which relates the Reynolds stress tensor to the velocity gradients and an eddy viscosity. The latter is obtained from the two transported scalars.

IV. Reynolds stress models. Here a transport equation is derived for the Reynolds tensor $\overline{u_i u_j}$. One transport equation has to be added for determining the length scale of the turbulence. Usually an equation for the dissipation ε is used.

Boussinesq Assumption

Boussinesq assumption, where the *Reynolds stress tensor* in the time averaged Navier-Stokes equation is replaced by *the turbulent viscosity multiplied by the velocity gradients.*

$$\rho\overline{u_i u_j} = -\mu_t(\bar{U}_{i,j} + \bar{U}_{j,i}).$$

$i = j$

$$\overline{u_i u_i} \equiv 2k$$

$$\rho\overline{u_i u_j} = -\mu_t(\bar{U}_{i,j} + \bar{U}_{j,i}) + \frac{2}{3}\delta_{ij}\rho k.$$

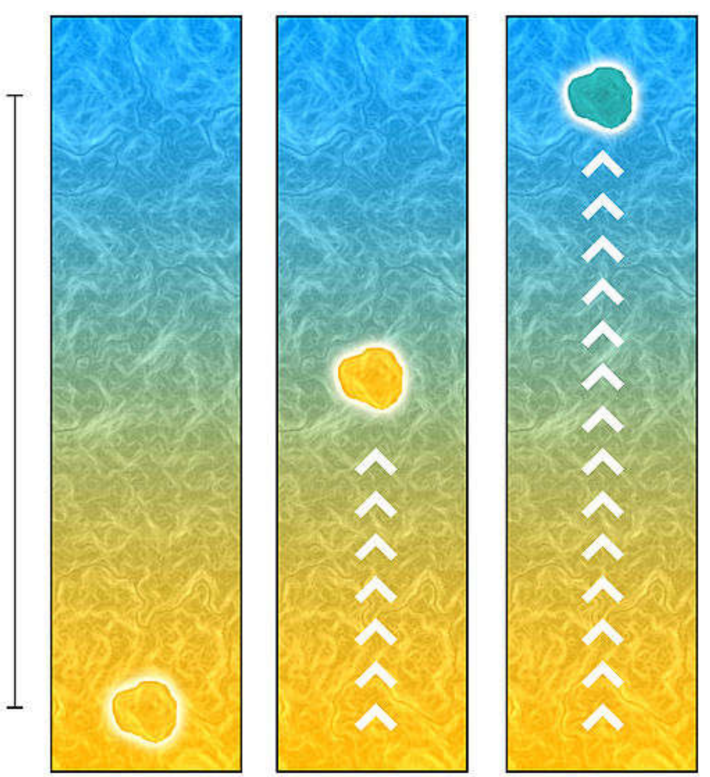
$$\delta_{ij\ if\ i\neq j} = 0$$

Note that contraction of δ_{ij} gives

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$$

Algebraic Models

MIXING LENGTH MODEL



a length called mixing length which is the average distance perpendicular to flow a small fluid mass will travel before its momentum is changed by new environment.

Fluid, which comes to the layer y_1 from a layer $(y_1 - l)$ has a positive value of v'. If the lump of fluid retains its original momentum then its velocity at its current location y_1 is smaller than the velocity prevailing there. The difference in velocities is then

$$\Delta u_1 = \bar{u}(y_1) - \bar{u}(y_1 - l) \approx l\left(\frac{\partial \bar{u}}{\partial y}\right)_{y1}$$

$$\Delta u_2 = \bar{u}(y_1 + l) - \bar{u}(y_1) \approx l\left(\frac{\partial \bar{u}}{\partial y}\right)_{y1}$$

$$\left|\overline{u'}\right| = \frac{1}{2}\left(\left|\Delta u_1\right| + \left|\Delta u_2\right|\right) = l\left|\left(\frac{\partial \bar{u}}{\partial y}\right)\right|_{y1}$$

$$\left|\overline{v'}\right| \sim \left|\overline{u'}\right|$$

or, $$\left|\overline{v'}\right| = (const)\left|\overline{u'}\right| = (const)\, l\left|\left(\frac{\partial \bar{u}}{\partial y}\right)\right|$$

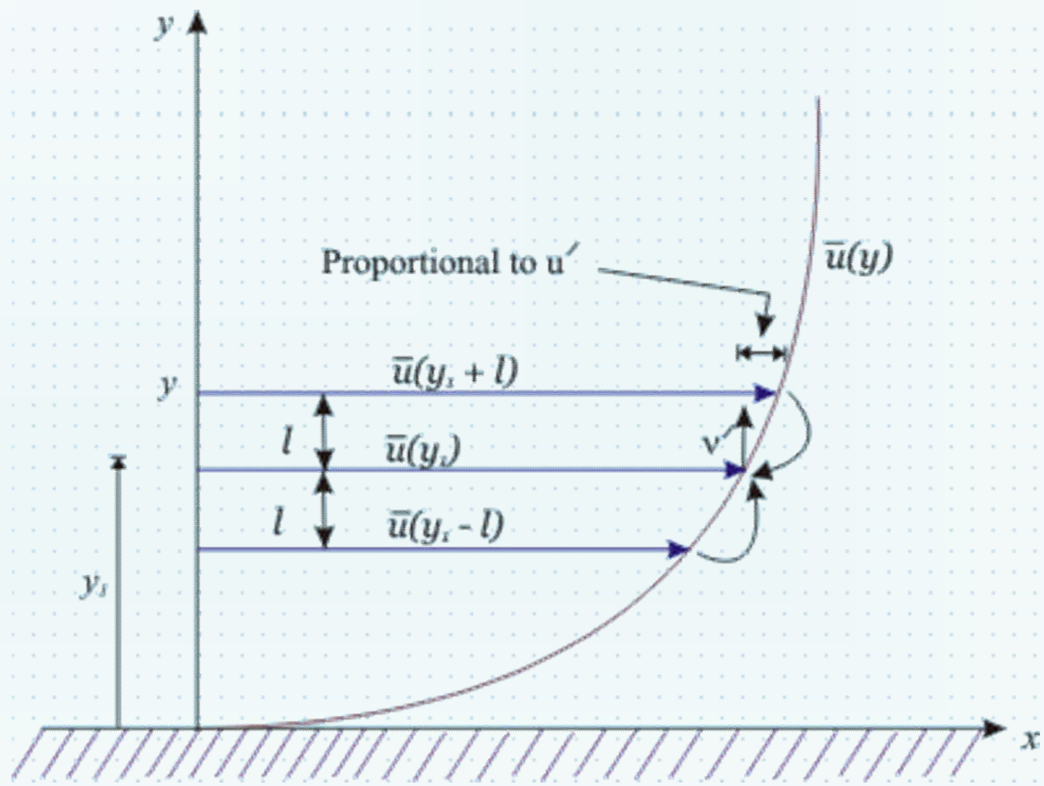
$$\overline{u'v'} = -C_1\left|\overline{u'}\right|\left|\overline{v'}\right|$$

$$\overline{u'v'} = -C_2 l^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2$$

$$\tau_t = -\rho\overline{u'v'} = \mu_t \frac{\partial \bar{u}}{\partial y}$$

$$\mu_t = \rho l^2\left|\frac{\partial \bar{u}}{\partial y}\right|$$

$$\nu_t = l^2\left|\frac{\partial \bar{u}}{\partial y}\right|$$



$$\nu_t = l^2 \left|\frac{\partial \bar{u}}{\partial y}\right|$$

$$\boxed{l = \kappa y}$$

κ is an empirical constant

for von Kármán's constant κ turns out to be 0.4

$$\nu_t = \kappa^2 y^2 \left|\frac{\partial \bar{u}}{\partial y}\right|$$

the van Driest (1956) model assumes

$$l_{\text{mix}} = ky\left[1 - \exp\left(\frac{y^+}{A_0^+}\right)\right]$$

where $A_0^+ = 26$, $y^+ = y\,u^*/\nu$, $u^* = \sqrt{(\tau_w/\rho)}$, τ_w is the wall shear stress (Smith and Cebeci, 1967; Baldwin and Lomax, 1978).

Cebeci-Smith Model

$$\mu_T = \begin{cases} \mu_{T_i}, & y \leq y_m \\ \mu_{T_o}, & y > y_m \end{cases}$$

where y_m is the smallest value of y for which $\mu_{T_i} = \mu_{T_o}$. The values of μ_T in the inner layer, μ_{T_i}, and the outer layer, μ_{T_o}, are computed as follows.

Inner Layer:

$$\mu_{T_i} = \rho \ell_{mix}^2 \left[\left(\frac{\partial U}{\partial y} \right)^2 + \left(\frac{\partial V}{\partial x} \right)^2 \right]^{1/2}$$

$$\ell_{mix} = \kappa y \left[1 - e^{-y^+/A^+} \right]$$

Outer layer:

$$\mu_{T_o} = \alpha \rho U_e \delta_v^* F_{Kleb}(y;\delta)$$

$$\kappa = 0.40, \qquad \alpha = 0.0168, \qquad A^+ = 26 \left[1 + y \frac{dP/dx}{\rho u_\tau^2} \right]^{-1/2}$$

F_{Kleb} is the Klebanoff intermittency function

$$F_{Kleb}(y;\delta) = \left[1 + 5.5 \left(\frac{y}{\delta} \right)^6 \right]^{-1}$$

$$\delta_v^* = \int_0^\delta (1 - U/U_e)\, dy$$

velocity thickness

U_e is boundary-layer edge velocity

Baldwin-Lomax Model

Inner Layer:

$$\mu_{T_i} = \rho \ell_{mix}^2 |\omega|$$

$$\ell_{mix} = \kappa y \left[1 - e^{-y^+/A_o^+} \right]$$

Outer Layer:

$$\mu_{T_o} = \rho \alpha C_{cp} F_{wake} F_{Kleb}(y; y_{max}/C_{Kleb})$$

$$F_{wake} = \min \left[y_{max} F_{max}; C_{wk} y_{max} U_{dif}^2 / F_{max} \right]$$

$$F_{max} = \frac{1}{\kappa} \left[\max_y (\ell_{mix} |\omega|) \right]$$

where y_{max} is the value of y at which $\ell_{mix}|\omega|$ achieves its maximum value.

$$\left. \begin{array}{lll} \kappa = 0.40, & \alpha = 0.0168, & A_o^+ = 26 \\ C_{cp} = 1.6, & C_{Kleb} = 0.3, & C_{wk} = 1 \end{array} \right\}$$

$$F_{Kleb}(y; \delta) = \left[1 + 5.5 \left(\frac{y}{\delta} \right)^6 \right]^{-1}$$

δ replaced by y_{max}/C_{Kleb}

$$\omega = \left[\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \right)^2 + \left(\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z} \right)^2 + \left(\frac{\partial U}{\partial z} - \frac{\partial W}{\partial x} \right)^2 \right]^{1/2}$$

ω is the magnitude of the vorticity

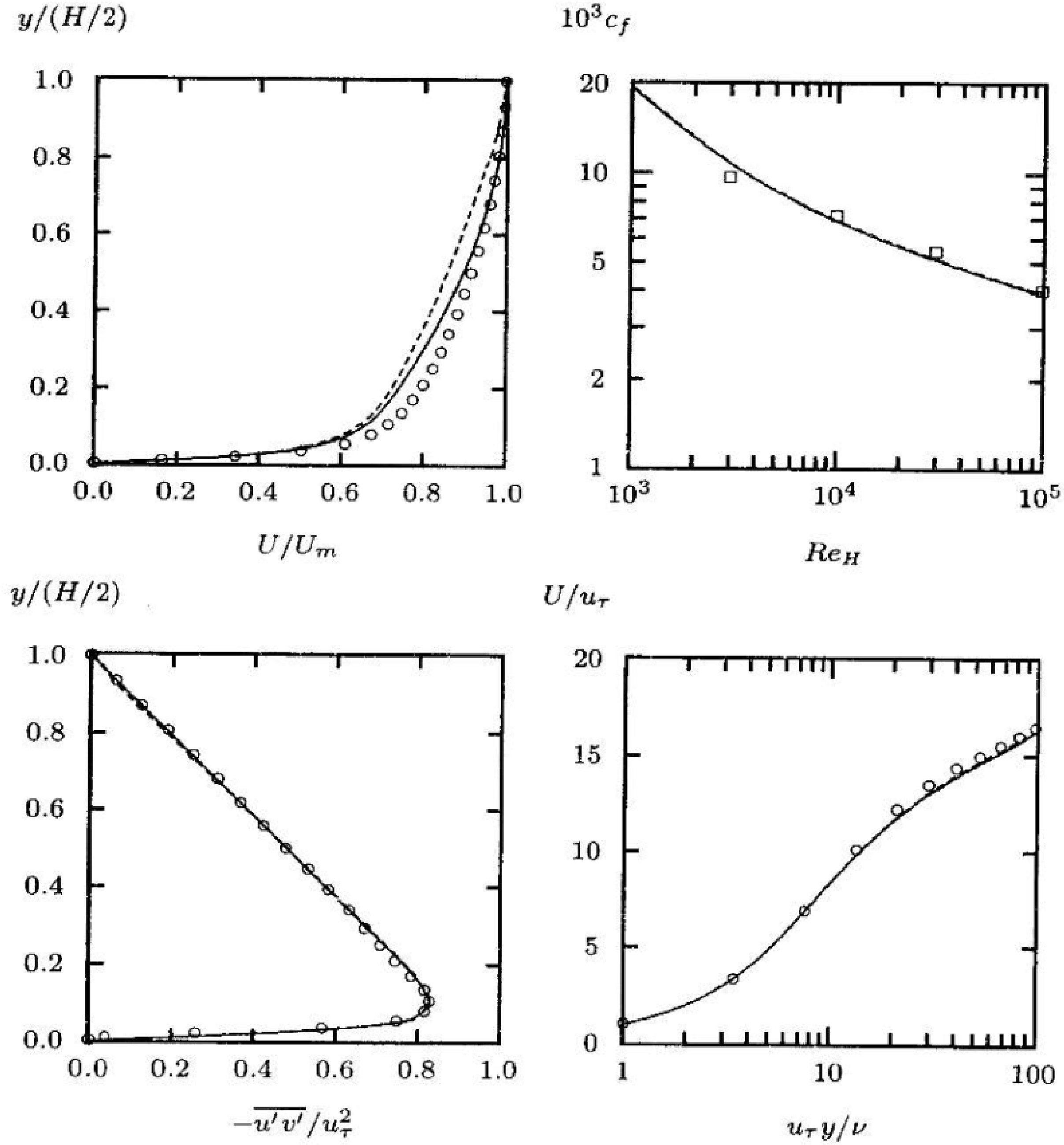


Figure 3.13: *Comparison of computed and measured channel-flow properties, Re_H = 13750. —— Baldwin-Lomax model; - - - Cebeci-Smith model; ○ Mansour et al. (DNS); □ Halleen-Johnston correlation.*

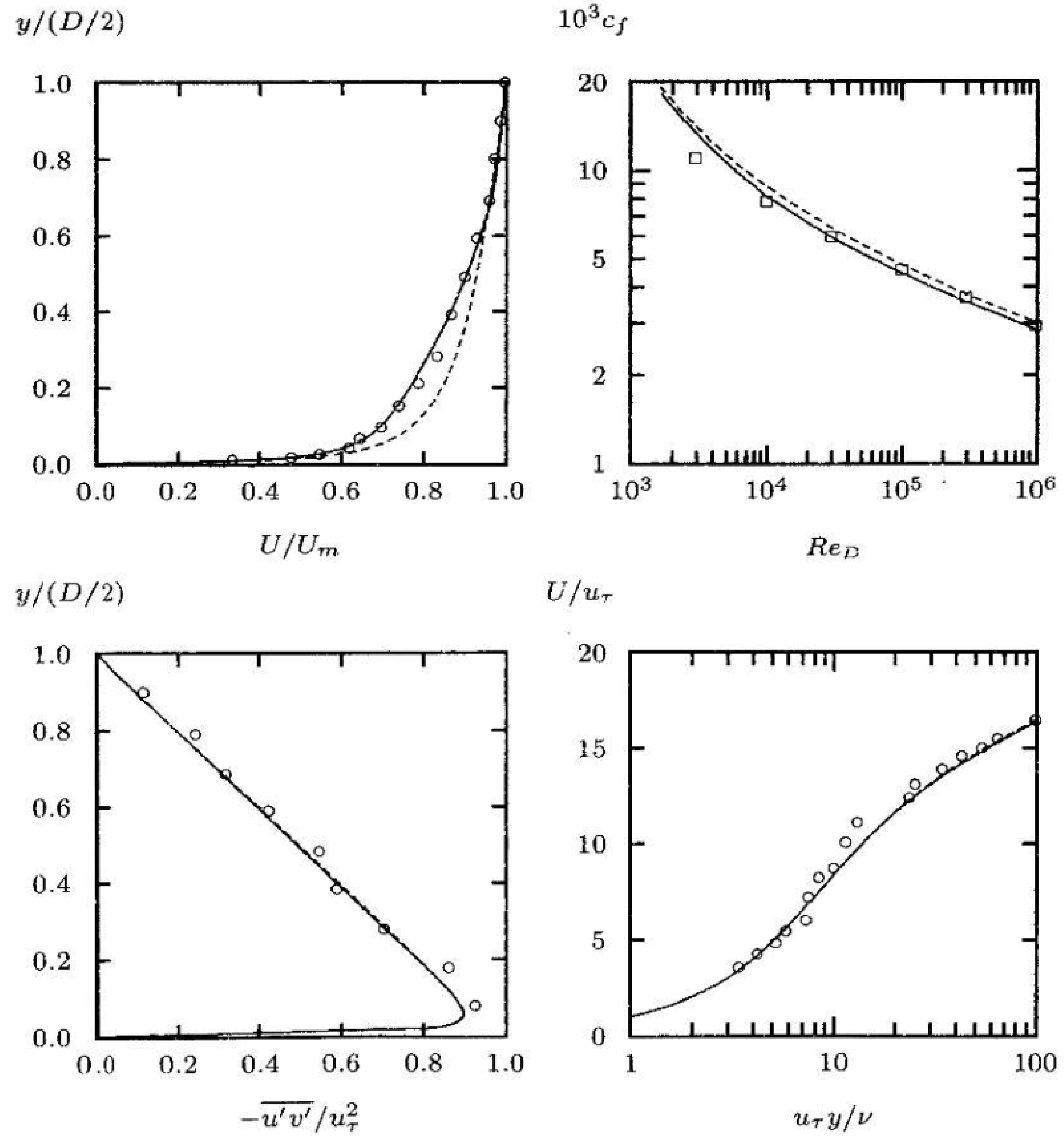


Figure 3.14: *Comparison of computed and measured pipe-flow properties, Re_D = 40000. —— Baldwin-Lomax model; - - - Cebeci-Smith model; ∘ Laufer; □ Prandtl correlation.*

Flat plate boundary layer flow

Table 3.1: *Differences Between Computed and Measured Skin Friction.*

Pressure Gradient	Flows	Baldwin-Lomax	Cebeci-Smith
Favorable	1400, 1300, 2700, 6300	7%	5%
Mild Adverse	1100, 2100, 2500, 4800	6%	7%
Moderate Adverse	2400, 2600, 3300, 4500	10%	15%
Strong Adverse	0141, 1200, 4400, 5300	14%	16%
All	–	9%	11%

The turbulent viscosity is estimated – using dimensional analysis – as the product of a turbulent velocity, $\mathcal{U}$, and length scale, $\mathcal{L}$,

$$\nu_t \propto \mathcal{U}\mathcal{L}$$

The velocity scale is taken as $k^{1/2}$ and the length scale as $k^{3/2}/\varepsilon$ which gives

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

$$C_\mu = 0.09.$$

$$\boxed{\nu_t \sim u \times l_m \sim k_t^{1/2} \times k_t^{3/2}/\epsilon^h \qquad \text{and hence,} \qquad \nu_t \equiv C_\mu \frac{k_t^2}{\epsilon^h}}$$