بسم الله الرحمن الرحيم

Prediction of Turbulent flow – Part 3

By: M. Farhadi

The turbulent viscosity is estimated – using dimensional analysis – as the product of a turbulent velocity, $\mathcal{U}$, and length scale, $\mathcal{L}$,

$$\nu_t \propto \mathcal{U}\mathcal{L}$$

The velocity scale is taken as $k^{1/2}$ and the length scale as $k^{3/2}/\varepsilon$ which gives

$$u \sim \sqrt{k_t},$$

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

$$\boxed{\nu_t \sim u \times l_m \sim k_t^{1/2} \times k_t^{3/2}/\epsilon^h \quad \text{and hence,} \quad \nu_t \equiv C_\mu \frac{k_t^2}{\epsilon^h}}$$

$$C_\mu = 0.09.$$

One-Equation Model

$$\frac{\partial k}{\partial t} + \underbrace{\bar{v}_j \frac{\partial k}{\partial x_j}}_{C^k} = \underbrace{-\overline{v_i' v_j'} \frac{\partial \bar{v}_i}{\partial x_j}}_{P^k} \underbrace{-\nu \overline{\frac{\partial v_i'}{\partial x_j} \frac{\partial v_i'}{\partial x_j}}}_{\varepsilon} \underbrace{-\frac{\partial}{\partial x_j}\left\{\overline{v_j'\left(\frac{p'}{\rho} + \frac{1}{2} v_i' v_i'\right)}\right\}}_{D_t^k} + \underbrace{\nu \frac{\partial^2 k}{\partial x_j \partial x_j}}_{D_\nu^k} \underbrace{-g_i \beta \overline{v_i' \theta'}}_{G^k}$$

$$C^k = P^k + D^k + G^k - \varepsilon$$

$$D^k = D_t^k + D_\nu^k$$

$$\frac{\partial k}{\partial t} + \underbrace{\bar{v}_j \frac{\partial k}{\partial x_j}}_{C^k} = \underbrace{-\overline{v_i'v_j'}\frac{\partial \bar{v}_i}{\partial x_j}}_{P^k} \underbrace{-\nu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}}_{\varepsilon} \underbrace{-\frac{\partial}{\partial x_j}\left\{\overline{v_j'\left(\frac{p'}{\rho}+\frac{1}{2}v_i'v_i'\right)}\right\}}_{D_t^k} + \underbrace{\nu\frac{\partial^2 k}{\partial x_j \partial x_j}}_{D_\nu^k} \underbrace{-g_i\beta\overline{v_i'\theta'}}_{G^k}$$

$$C^k = P^k + D^k + G^k - \varepsilon \qquad D^k = D_t^k + D_\nu^k$$

$$\overline{v_i'\theta'} = -\alpha_t \frac{\partial \bar{\theta}}{\partial x_i} \qquad \alpha_t = \frac{\nu_t}{\sigma_\theta} \qquad 0.7 \le \sigma_\theta \le 0.9.$$

turbulent Prandtl number

$$-\overline{v_i'v_j'} = \nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)$$

$$\overline{v_i'v_j'} = -\nu_t\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) + \frac{1}{3}\delta_{ij}\overline{v_k'v_k'} = -2\nu_t\bar{s}_{ij} + \frac{2}{3}\delta_{ij}k$$

$$P^k = \left\{\nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij} k\right\} \frac{\partial \bar{v}_i}{\partial x_j}$$

$$= \nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right) \frac{\partial \bar{v}_i}{\partial x_j} = \nu_t 2\bar{s}_{ij}(\bar{s}_{ij} + \Omega_{ij}) = 2\nu_t \bar{s}_{ij}\bar{s}_{ij}$$

$$\bar{s}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right), \quad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial \bar{v}_j}{\partial x_i}\right), \quad \frac{\partial \bar{v}_i}{\partial x_j} = \bar{s}_{ij} + \Omega_{ij}$$

the fact that $\bar{s}_{ij}\Omega_{ij} = 0$ $\partial \bar{v}_i / \partial x_i = 0$

$$\frac{\partial k}{\partial t} + \underbrace{\bar{v}_j \frac{\partial k}{\partial x_j}}_{C^k} = \underbrace{-\overline{v_i' v_j'} \frac{\partial \bar{v}_i}{\partial x_j}}_{P^k} \underbrace{- \nu \overline{\frac{\partial v_i'}{\partial x_j} \frac{\partial v_i'}{\partial x_j}}}_{\varepsilon} \underbrace{- \frac{\partial}{\partial x_j} \left\{ \overline{v_j' \left(\frac{p'}{\rho} + \frac{1}{2} v_i' v_i' \right)} \right\}}_{D_t^k} + \underbrace{\nu \frac{\partial^2 k}{\partial x_j \partial x_j}}_{D_\nu^k} \underbrace{- g_i \beta \overline{v_i' \theta'}}_{G^k}$$

$$\underbrace{-\frac{\partial}{\partial x_j} \left\{ \overline{v_j' \left(\frac{p'}{\rho} + \frac{1}{2} v_i' v_i' \right)} \right\}}_{D_t^k} = \frac{\mu_t}{\sigma_k} \frac{\partial k}{\partial x_j}$$

σ_k is the turbulent Prandtl number for k.

There is no model for the pressure diffusion term. It is small and thus it is simply neglected

$$D^k = \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right]$$

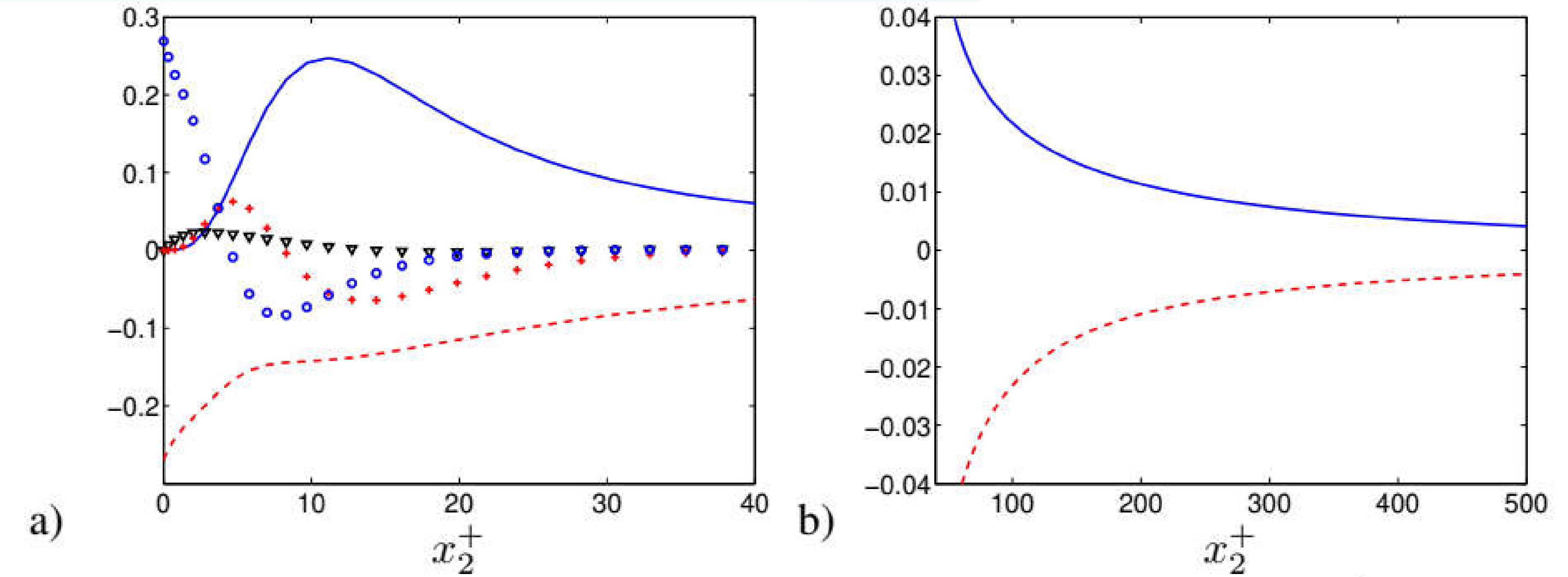

Figure 8.3: Channel flow at $Re_\tau = 2000$. Terms in the k equation scaled by u_τ^4/ν. $Re_\tau = 2000$. a) Zoom near the wall; b) Outer region. DNS (Direct Numerical Simulation) data [15, 16]. ▬ : P^k; ‒ ‒ : $-\varepsilon$; ∇: $-\partial \overline{v' p'}/\partial x_2$; $+$: $-\partial \overline{v_2' v_i' v_i'/2}/\partial x_2$; $\circ$: $\nu \partial^2 k / \partial x_2^2$.

The dissipation term ε_{ij} is active for the small-scale turbulence. Because of the cascade process and vortex stretching the small scale turbulence is isotropic. This means that the velocity fluctuations of the small scale turbulence have no preferred direction. This gives:

1. $\overline{v_1'^2} = \overline{v_2'^2} = \overline{v_3'^2}$.
2. All shear stresses are zero, i.e. $\overline{v_i' v_j'} = 0 \quad \text{if} \quad i \neq j$

because the fluctuations in two different coordinate directions are not correlated.

$$\overline{\frac{\partial v_1'}{\partial x_k}\frac{\partial v_1'}{\partial x_k}} = \overline{\frac{\partial v_2'}{\partial x_k}\frac{\partial v_2'}{\partial x_k}} = \overline{\frac{\partial v_3'}{\partial x_k}\frac{\partial v_3'}{\partial x_k}}$$

$$\overline{\frac{\partial v_i'}{\partial x_k}\frac{\partial v_j'}{\partial x_k}} = 0 \quad \text{if} \quad i \neq j$$

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij}$$

$$\varepsilon = \tfrac{1}{2}\varepsilon_{ii}$$

$$\varepsilon = O\left(\frac{u^3}{\ell}\right)$$

$$u = \sqrt{k}$$

$$\varepsilon \sim \frac{k^{\frac{3}{2}}}{\ell}$$

$$\varepsilon = \nu \overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}$$

$$\epsilon = C_D k^{3/2}/\ell$$

Turbulence length scale

$$0.07 < C_D < 0.09$$

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$$\frac{\partial k}{\partial t} + \bar{v}_j \frac{\partial k}{\partial x_j} = \nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) \frac{\partial \bar{v}_i}{\partial x_j} + g_i \beta \frac{\nu_t}{\sigma_\theta} \frac{\partial \bar{\theta}}{\partial x_i} - \varepsilon + \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right]$$

Turbulence length scale

$$\epsilon = C_D k^{3/2} / \ell$$

$$0.07 < C_D < 0.09$$

$$\nu_t = c_\mu \frac{k^2}{\varepsilon}$$

The unknown turbulent length scale must be given, and often an algebraic expression is used. This length scale is, for example, taken as proportional to the thickness of the boundary layer, the width of a jet or a wake. The main disadvantage of this type of model is that it is not applicable to general flows since it is not possible to find a general expression for an algebraic length scale.

Kinematic Eddy Viscosity:

$$\nu_T = C_\mu \nu \tilde{R}_T D_1 D_2$$

Turbulence Reynolds Number:

$$\frac{\partial}{\partial t}\left(\nu \tilde{R}_T\right) + U_j \frac{\partial}{\partial x_j}\left(\nu \tilde{R}_T\right) = \left(C_{\epsilon 2} f_2 - C_{\epsilon 1}\right)\sqrt{\nu \tilde{R}_T P}$$

$$+(\nu + \nu_T/\sigma_\epsilon)\frac{\partial^2(\nu \tilde{R}_T)}{\partial x_k \partial x_k} - \frac{1}{\sigma_\epsilon}\frac{\partial \nu_T}{\partial x_k}\frac{\partial(\nu \tilde{R}_T)}{\partial x_k}$$

Closure Coefficients and Auxiliary Relations:

$$C_{\epsilon 1} = 1.2, \quad C_{\epsilon 2} = 2.0, \quad C_\mu = 0.09, \quad A_o^+ = 26, \quad A_2^+ = 10$$

$$\frac{1}{\sigma_\epsilon} = (C_{\epsilon 2} - C_{\epsilon 1})\frac{\sqrt{C_\mu}}{\kappa^2}, \qquad \kappa = 0.41$$

$$P = \nu_T \left[\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - \frac{2}{3}\frac{\partial U_k}{\partial x_k}\frac{\partial U_k}{\partial x_k}\right]$$

$$D_1 = 1 - e^{-y^+/A_o^+} \quad \text{and} \quad D_2 = 1 - e^{-y^+/A_2^+}$$

$$f_2 = \frac{C_{\epsilon 1}}{C_{\epsilon 2}} + \left(1 - \frac{C_{\epsilon 1}}{C_{\epsilon 2}}\right)\left(\frac{1}{\kappa y^+} + D_1 D_2\right) \cdot$$

$$\left[\sqrt{D_1 D_2} + \frac{y^+}{\sqrt{D_1 D_2}}\left(\frac{D_2}{A_o^+}e^{-y^+/A_o^+} + \frac{D_1}{A_2^+}e^{-y^+/A_2^+}\right)\right]$$

Spalart and Allmaras (1992)

Kinematic Eddy Viscosity:

$$\nu_T = \tilde{\nu} f_{v1}$$

Eddy Viscosity Equation:

$$\frac{\partial \tilde{\nu}}{\partial t} + U_j \frac{\partial \tilde{\nu}}{\partial x_j} = c_{b1} \tilde{S} \tilde{\nu} - c_{w1} f_w \left(\frac{\tilde{\nu}}{d} \right)^2 + \frac{1}{\sigma} \frac{\partial}{\partial x_k} \left[(\nu + \tilde{\nu}) \frac{\partial \tilde{\nu}}{\partial x_k} \right] + \frac{c_{b2}}{\sigma} \frac{\partial \tilde{\nu}}{\partial x_k} \frac{\partial \tilde{\nu}}{\partial x_k}$$

Closure Coefficients and Auxiliary Relations:

$$c_{b1} = 0.1355, \quad c_{b2} = 0.622, \quad c_{v1} = 7.1, \quad \sigma = 2/3$$

$$c_{w1} = \frac{c_{b1}}{\kappa^2} + \frac{(1 + c_{b2})}{\sigma}, \quad c_{w2} = 0.3, \quad c_{w3} = 2, \quad \kappa = 0.41$$

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}, \qquad f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}, \qquad f_w = g \left[\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6} \right]^{1/6}$$

$$\chi = \frac{\tilde{\nu}}{\nu}, \qquad g = r + c_{w2}(r^6 - r), \qquad r = \frac{\tilde{\nu}}{\tilde{S} \kappa^2 d^2}$$

$$\tilde{S} = S + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2}, \qquad S = \sqrt{2 \Omega_{ij} \Omega_{ij}}$$

The tensor $\Omega_{ij} = \frac{1}{2}(\partial U_i / \partial x_j - \partial U_j / \partial x_i)$ is the rotation tensor and d is distance from the closest surface.

Original model

The turbulent eddy viscosity is given by

$$\nu_t = \tilde{\nu} f_{v1}, \quad f_{v1} = \frac{\chi^3}{\chi^3 + C_{v1}^3}, \quad \chi := \frac{\tilde{\nu}}{\nu}$$

$$\frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} = C_{b1}[1 - f_{t2}]\tilde{S}\tilde{\nu} + \frac{1}{\sigma}\{\nabla \cdot [(\nu + \tilde{\nu})\nabla \tilde{\nu}] + C_{b2}|\nabla \tilde{\nu}|^2\} - \left[C_{w1} f_w - \frac{C_{b1}}{\kappa^2} f_{t2}\right] \left(\frac{\tilde{\nu}}{d}\right)^2 + f_{t1}\Delta U^2$$

$$\tilde{S} \equiv S + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2}, \quad f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}} \qquad S = \equiv \sqrt{2\Omega_{ij}\Omega_{ij}}$$

$$\Omega_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$f_w = g \left[\frac{1 + C_{w3}^6}{g^6 + C_{w3}^6}\right]^{1/6}, \quad g = r + C_{w2}(r^6 - r), \quad r \equiv \frac{\tilde{\nu}}{\tilde{S}\kappa^2 d^2}$$

$$f_{t1} = C_{t1} g_t \exp\left(-C_{t2} \frac{\omega_t^2}{\Delta U^2}[d^2 + g_t^2 d_t^2]\right)$$

$$f_{t2} = C_{t3} \exp(-C_{t4}\chi^2)$$

d is the distance to the closest surface

The constants are

σ	=	$2/3$
C_{b1}	=	0.1355
C_{b2}	=	0.622
κ	=	0.41
C_{w1}	=	$C_{b1}/\kappa^2 + (1 + C_{b2})/\sigma$
C_{w2}	=	0.3
C_{w3}	=	2
C_{v1}	=	7.1
C_{t1}	=	1
C_{t2}	=	2
C_{t3}	=	1.1
C_{t4}	=	2

The $k-\varepsilon$ model

The k equation

$$C^k = P^k + D^k + G^k - \varepsilon$$

$$\frac{\partial k}{\partial t} + \bar{v}_j \frac{\partial k}{\partial x_j} = \nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) \frac{\partial \bar{v}_i}{\partial x_j} + g_i \beta \frac{\nu_t}{\sigma_\theta} \frac{\partial \bar{\theta}}{\partial x_i}$$
$$-\varepsilon + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right]$$

The ε equation

$$C^\varepsilon = P^\varepsilon + D^\varepsilon + G^\varepsilon - \Psi^\varepsilon$$

$$\left.\begin{array}{ll} \partial k/\partial t & [m^2/s^3] \\ \partial \varepsilon/\partial t & [m^2/s^4] \end{array}\right\} \varepsilon/k = [1/s] \Longrightarrow P^\varepsilon + G^\varepsilon - \Psi^\varepsilon = \frac{\varepsilon}{k}\left(c_{\varepsilon 1} P^k + c_{\varepsilon 1} G^k - c_{\varepsilon 2} \varepsilon \right)$$

$$D^\varepsilon = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} \right]$$

The final form of the ε transport equation reads

$$\frac{\partial \varepsilon}{\partial t} + \bar{v}_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\varepsilon}{k}(c_{\varepsilon 1} P^k + c_{\varepsilon 1} G^k - c_{\varepsilon 2}\varepsilon) + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]$$

$$\begin{aligned}\frac{\partial \varepsilon}{\partial t} + \bar{v}_j \frac{\partial \varepsilon}{\partial x_j} &= \frac{\varepsilon}{k} c_{\varepsilon 1} \nu_t \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)\frac{\partial \bar{v}_i}{\partial x_j} \\ &\quad + c_{\varepsilon 1} g_i \frac{\varepsilon}{k}\frac{\nu_t}{\sigma_\theta}\frac{\partial \bar{\theta}}{\partial x_i} - c_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]\end{aligned}$$

$$\nu_t = c_\mu \frac{k^2}{\varepsilon}$$

$$(c_\mu, c_{\varepsilon 1}, c_{\varepsilon 2}, \sigma_k, \sigma_\varepsilon) = (0.09, 1.44, 1.92, 1, 1.3)$$

Knowledge of k and ϵ provides an estimation of the turbulent velocity scale $u' = \sqrt{2k/3}$ and of an integral length scale from $L_f \simeq u'^3/\epsilon$. Moreover, from the expression of the turbulent viscosity $\nu_t = u' \times l_m = C_\mu k^2/\epsilon$, the mixing length is given by $l_m \simeq \sqrt{3/2} C_\mu k^{3/2}/\epsilon$.

The $k-\omega$ Model

specific dissipation rate, ω (time)$^{-1}$

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$$\nu_T \sim k/\omega, \qquad \ell \sim k^{1/2}/\omega, \qquad \epsilon \sim \omega k$$

$\omega \propto \varepsilon / k$

Wilcox (2006) k-ω model.

Kinematic Eddy Viscosity:

$$\nu_T = \frac{k}{\tilde{\omega}}, \qquad \tilde{\omega} = \max\left\{\omega,\ C_{lim}\sqrt{\frac{2S_{ij}S_{ij}}{\beta^*}}\right\}, \qquad C_{lim} = \frac{7}{8}$$

Turbulence Kinetic Energy:

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \tau_{ij}\frac{\partial U_i}{\partial x_j} - \beta^* k\omega + \frac{\partial}{\partial x_j}\left[\left(\nu + \sigma^* \frac{k}{\omega}\right)\frac{\partial k}{\partial x_j}\right]$$

Specific Dissipation Rate:

$$\frac{\partial \omega}{\partial t} + U_j \frac{\partial \omega}{\partial x_j} = \alpha \frac{\omega}{k}\tau_{ij}\frac{\partial U_i}{\partial x_j} - \beta\omega^2 + \frac{\sigma_d}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\left(\nu + \sigma\frac{k}{\omega}\right)\frac{\partial \omega}{\partial x_j}\right]$$

Closure Coefficients and Auxiliary Relations:

$$\alpha = \frac{13}{25}, \quad \beta = \beta_o f_\beta, \quad \beta^* = \frac{9}{100}, \quad \sigma = \frac{1}{2}, \quad \sigma^* = \frac{3}{5}, \quad \sigma_{do} = \frac{1}{8}$$

$$\sigma_d = \begin{cases} 0, & \dfrac{\partial k}{\partial x_j}\dfrac{\partial \omega}{\partial x_j} \le 0 \\ \sigma_{do}, & \dfrac{\partial k}{\partial x_j}\dfrac{\partial \omega}{\partial x_j} > 0 \end{cases}$$

$$\beta_o = 0.0708, \quad f_\beta = \frac{1+85\chi_\omega}{1+100\chi_\omega}, \quad \chi_\omega \equiv \left| \frac{\Omega_{ij}\Omega_{jk}S_{ki}}{(\beta^*\omega)^3} \right|$$

$$\epsilon = \beta^* \omega k \qquad \text{and} \qquad \ell = k^{1/2}/\omega$$

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right), \qquad S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

Rotta's (1968) k-$k\ell$ model

integral length scale, ℓ,

$$R_{ij}(\mathbf{x},t;\mathbf{r}) = \overline{u_i'(\mathbf{x},t)\ u_j'(\mathbf{x}+\mathbf{r},t)}$$

$$k = \frac{1}{2}R_{ii}(\mathbf{x},t;\mathbf{0}) \qquad \text{and} \qquad k\ell = \frac{3}{16}\int_{-\infty}^{\infty} R_{ii}(\mathbf{x},t;r)\ dr$$

$$\frac{\partial}{\partial t}(k\ell) + U_j\frac{\partial}{\partial x_j}(k\ell) = C_{L1}\ell\tau_{ij}\frac{\partial U_i}{\partial x_j} - C_{L2}k^{3/2} + \frac{\partial}{\partial x_j}\left[\nu\frac{\partial}{\partial x_j}(k\ell) + (\nu_T/\sigma_{L1})\ell\frac{\partial k}{\partial x_j} + (\nu_T/\sigma_{L2})k\frac{\partial \ell}{\partial x_j}\right]$$

$$C_{L1} = 0.98, \quad C_{L2} = 0.059 + 702(\ell/y)^6, \quad C_D = 0.09, \quad \sigma_k = \sigma_{L1} = \sigma_{L2} = 1$$

The k-kL-MEAH2015 two-equation turbulence model

$$\frac{\partial}{\partial t}\rho \mathrm{k} + \nabla\cdot(\rho\vec{\mathrm{U}}\mathrm{k}) = \mathrm{P_k} - \mathrm{C_k}\rho\frac{\mathrm{k}^{5/2}}{(\mathrm{kL})} - 2\mu\frac{\mathrm{k}}{\mathrm{d}^2} + \nabla\cdot[(\mu+\sigma_{\mathrm{k}}\mu_{\mathrm{t}})\nabla \mathrm{k}]$$

$$\frac{\partial}{\partial t}\rho(\mathrm{kL}) + \nabla\cdot(\rho\vec{\mathrm{U}}(\mathrm{kL})) = \mathrm{C}_{\phi_1}\frac{(\mathrm{kL})}{\mathrm{k}}\mathrm{P_k} - \mathrm{C}_{\phi_2}\rho\mathrm{k}^{3/2} - 6\mu\frac{(\mathrm{kL})}{\mathrm{d}^2}f_\Phi + \nabla\cdot[(\mu+\sigma_\phi\mu_{\mathrm{t}})\nabla(\mathrm{kL})]$$

$$\mu_{\mathrm{t}} = \mathrm{C}_\mu{}^{1/4}\frac{\rho(\mathrm{kL})}{\mathrm{k}^{1/2}}$$

$$\mathrm{P} = \tau_{ij}\frac{\partial u_i}{\partial x_j},\quad \tau_{ij} = 2\mu_{\mathrm{t}}\left(\mathrm{S}_{ij} - \frac{1}{3}tr\{\boldsymbol{S}\}\delta_{ij}\right) - \frac{2}{3}\rho\mathrm{k}\delta_{ij},\quad \mathrm{S}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\mathrm{P_k} = \min\left(\mathrm{P}, 20\,\mathrm{C}_\mu{}^{3/4}\frac{\rho\mathrm{k}^{5/2}}{(\mathrm{kL})}\right)$$

$$\mathrm{C_k} = \mathrm{C}_\mu{}^{3/4},\quad \mathrm{C}_{\phi_1} = \zeta_1 - \zeta_2\left(\frac{(\mathrm{kL})}{\mathrm{kL}_{vk}}\right)^2,\quad \mathrm{C}_{\phi_2} = \zeta_3$$

$$f_\Phi = \frac{1+\mathrm{C}_{d1}\xi}{1+\xi^4},\quad \xi = \frac{\rho\sqrt{0.3\mathrm{kd}}}{20\mu}$$

$$\mathrm{L}_{vk} = \kappa\left|\frac{\mathrm{U}'}{\mathrm{U}''}\right|,\quad \mathrm{U}' = \sqrt{2\mathrm{S}_{ij}\mathrm{S}_{ij}},\quad \mathrm{U}'' = \sqrt{\frac{\partial^2\mathrm{u}_i}{\partial x_k^2}\frac{\partial^2\mathrm{u}_i}{\partial x_j^2}}$$

$$\sigma_{\mathrm{k}} = 1.0,\quad \sigma_{(\mathrm{kL})} = 1.0$$
$$\kappa = 0.41,\quad \mathrm{C}_\mu = 0.09$$
$$\zeta_1 = 1.2,\quad \zeta_2 = 0.97,\quad \zeta_3 = 0.13$$
$$\mathrm{C}_{11} = 10.0,\quad \mathrm{C}_{12} = 1.3,\quad \mathrm{C}_{d1} = 4.7$$

A limiter is applied on L_{vk},

$$\mathrm{L}_{vk,\min} \le \mathrm{L}_{vk} \le \mathrm{L}_{vk,\max},\quad \mathrm{L}_{vk,\min} = \frac{(\mathrm{kL})}{\mathrm{kC}_{11}},\quad \mathrm{L}_{vk,\max} = \mathrm{C}_{12}\kappa\mathrm{d}f_{\mathrm{p}}$$

$$f_{\mathrm{p}} = min\left[max\left(\frac{\mathrm{P_k}(\mathrm{kL})}{\mathrm{C}^{3/4}\rho\mathrm{k}^{5/2}}, 0.5\right), 1.0\right]$$

$$\mathrm{k_{wall}} = (\mathrm{kL})_{\mathrm{wall}} = 0,\quad \mathrm{k}_\infty = 9\times10^{-10}\mathrm{a}_\infty^2,\quad (\mathrm{kL})_\infty = 1.5589\times10^{-6}\frac{\mu_\infty\mathrm{a}_\infty}{\rho_\infty}$$

Zeierman and Wolfshtein (1986)

Zeierman-Wolfshtein k-$k\tau$ model

autocorrelation tensor defined $\mathcal{R}_{ij}(\mathbf{x},t;t') = \overline{u_i'(\mathbf{x},t)u_j'(\mathbf{x},t+t')}$

The turbulence kinetic energy is half the trace of $\mathcal{R}_{ij}$ with $t'=0$, while the **integral time scale** is proportional to the integral of $\mathcal{R}_{ii}$ over all possible values of t'. Thus,

$$k = \frac{1}{2}\mathcal{R}_{ii}(\mathbf{x},t;0) \quad \text{and} \quad k\tau = \frac{1}{2}\int_0^\infty \mathcal{R}_{ii}(\mathbf{x},t;t')\,dt'$$

Kinematic Eddy Viscosity:

$$\nu_T = C_\mu k\tau$$

Turbulence Kinetic Energy:

$$\frac{\partial k}{\partial t} + U_j\frac{\partial k}{\partial x_j} = \tau_{ij}\frac{\partial U_i}{\partial x_j} - \frac{k}{\tau} + \frac{\partial}{\partial x_j}\left[(\nu + \nu_T/\sigma_k)\frac{\partial k}{\partial x_j}\right]$$

Integral Time Scale:

$$\frac{\partial}{\partial t}(k\tau) + U_j\frac{\partial}{\partial x_j}(k\tau) = C_{\tau 1}\tau\tau_{ij}\frac{\partial U_i}{\partial x_j} - C_{\tau 2}k + \frac{\partial}{\partial x_j}\left[(\nu + \nu_T/\sigma_\tau)\frac{\partial}{\partial x_j}(k\tau)\right]$$

$C_{\tau 1} = 0.173, \quad C_{\tau 2} = 0.225, \quad C_\mu = 0.09, \quad \sigma_k = 1.46, \quad \sigma_\tau = 10.8$

$\omega = 1/(C_\mu\tau), \quad \epsilon = k/\tau \quad \text{and} \quad \ell = C_\mu k^{1/2}\tau$

Speziale, Abid and Anderson (1990) $k-\tau$ Model

$\epsilon = k/\tau$

the turbulent time scale τ

Kinematic Eddy Viscosity:

$$\nu_T = C_\mu k \tau$$

Turbulence Kinetic Energy:

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \tau_{ij} \frac{\partial U_i}{\partial x_j} - \frac{k}{\tau} + \frac{\partial}{\partial x_j}\left[(\nu + \nu_T/\sigma_k)\frac{\partial k}{\partial x_j}\right]$$

τ equation

$$\begin{aligned}\frac{\partial \tau}{\partial t} + U_j \frac{\partial \tau}{\partial x_j} = (1 - C_{\epsilon 1})\frac{\tau}{k}\tau_{ij}\frac{\partial U_i}{\partial x_j} + (C_{\epsilon 2} - 1) \\ + \frac{\partial}{\partial x_j}\left[(\nu + \nu_T/\sigma_{\tau 2})\frac{\partial \tau}{\partial x_j}\right] \\ + \frac{2}{k}(\nu + \nu_T/\sigma_{\tau 1})\frac{\partial k}{\partial x_k}\frac{\partial \tau}{\partial x_k} - \frac{2}{\tau}(\nu + \nu_T/\sigma_{\tau 2})\frac{\partial \tau}{\partial x_k}\frac{\partial \tau}{\partial x_k}\end{aligned}$$

$$C_{\epsilon 1} = 1.44, \quad C_{\epsilon 2} = 1.83, \quad C_\mu = 0.09, \quad \sigma_k = \sigma_{\tau 1} = \sigma_{\tau 2} = 1.36$$

Low-Re Number Turbulence Models

We start by studying how various quantities behave close to the wall when $x_2 \to 0$. Taylor expansion of the fluctuating velocities v_i' (also valid for the mean velocities $\bar{v}_i$) gives

$$
\begin{aligned}
v_1' &= a_0 + a_1 x_2 + a_2 x_2^2 + \dots \\
v_2' &= b_0 + b_1 x_2 + b_2 x_2^2 + \dots \\
v_3' &= c_0 + c_1 x_2 + c_2 x_2^2 + \dots
\end{aligned}
$$

no-slip conditions $v_1' = v_2' = v_3' = 0$ which gives $a_0 = b_0 = c_0$

at the wall $\partial v_1'/\partial x_1 = \partial v_3'/\partial x_3 = 0$ so that

the continuity equation gives $\partial v_2'/\partial x_2 = 0$. $b_1 = 0$.

$$
\begin{aligned}
v_1' &= a_1 x_2 &+\quad& a_2 x_2^2 + \dots \\
v_2' &= && b_2 x_2^2 + \dots \\
v_3' &= c_1 x_2 &+\quad& c_2 x_2^2 + \dots
\end{aligned}
$$

$$
\begin{aligned}
\overline{v_1'^2} &= \overline{a_1^2} x_2^2 + \dots &&= \mathcal{O}(x_2^2) \\
\overline{v_2'^2} &= \overline{b_2^2} x_2^4 + \dots &&= \mathcal{O}(x_2^4) \\
\overline{v_3'^2} &= \overline{c_1^2} x_2^2 + \dots &&= \mathcal{O}(x_2^2) \\
\overline{v_1' v_2'} &= \overline{a_1 b_2} x_2^3 + \dots &&= \mathcal{O}(x_2^3) \\
k &= (\overline{a_1^2} + \overline{c_1^2}) x_2^2 + \dots &&= \mathcal{O}(x_2^2) \\
\partial \bar{v}_1/\partial x_2 &= \overline{a_1} + \dots &&= \mathcal{O}(x_2^0) \\
\partial v_1'/\partial x_2 &= \overline{a_1} + \dots &&= \mathcal{O}(x_2^0) \\
\partial v_2'/\partial x_2 &= 2\overline{b_2} x_2 + \dots &&= \mathcal{O}(x_2^1) \\
\partial v_3'/\partial x_2 &= \overline{a_1} + \dots &&= \mathcal{O}(x_2^0)
\end{aligned}
$$

$$
\frac{\partial \rho \bar{v}_1 k}{\partial x_1} + \frac{\partial \rho \bar{v}_2 k}{\partial x_2} = \underbrace{-\rho \overline{v_1' v_2'} \frac{\partial \bar{v}_1}{\partial x_2}}_{\mathcal{O}(x_2^3)} - \frac{\partial \overline{p' v_2'}}{\partial x_2} \underbrace{- \frac{\partial}{\partial x_2}\left(\frac{1}{2} \rho \overline{v_2' v_i' v_i'}\right)}_{\mathcal{O}(x_2^3)} + \mu \frac{\partial^2 k}{\partial x_2^2} - \underbrace{\mu \overline{\frac{\partial v_i'}{\partial x_j} \frac{\partial v_i'}{\partial x_j}}}_{\mathcal{O}(x_2^0)}
$$

$$\frac{\partial\rho\bar{v}_1 k}{\partial x_1}+\frac{\partial\rho\bar{v}_2 k}{\partial x_2} = \underbrace{\mu_t\left(\frac{\partial \bar{v}_1}{\partial x_2}\right)^2}_{\mathcal{O}(x_2^4)} + \underbrace{\frac{\partial}{\partial x_2}\left(\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_2}\right)}_{\mathcal{O}(x_2^4)} + \mu\frac{\partial^2 k}{\partial x_2^2} - \underbrace{\rho\varepsilon}_{\mathcal{O}(x_2^0)}$$

$$\frac{\partial\rho\bar{v}_1 k}{\partial x_1}+\frac{\partial\rho\bar{v}_2 k}{\partial x_2} = \underbrace{-\rho\overline{v_1'v_2'}\frac{\partial\bar{v}_1}{\partial x_2}}_{\mathcal{O}(x_2^3)} - \frac{\partial\overline{p'v_2'}}{\partial x_2} \underbrace{- \frac{\partial}{\partial x_2}\left(\frac{1}{2}\rho\overline{v_2'v_i'v_i'}\right)}_{\mathcal{O}(x_2^3)} + \mu\frac{\partial^2 k}{\partial x_2^2} - \underbrace{\mu\overline{\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}}}_{\mathcal{O}(x_2^0)}$$

$$\nu_t = C_\mu\frac{k^2}{\varepsilon} = \frac{\mathcal{O}(x_2^4)}{\mathcal{O}(x_2^0)} = \mathcal{O}(x_2^4)$$

replacing the C_μ constant by $C_\mu f_\mu$ where f_μ is a damping function f_μ so that $f_\mu = \mathcal{O}(x_2^{-1})$

when $x_2 \to 0$ and $f_\mu \to 1$ when $x_2^+ \geq 50$. Now we get $\nu_t = \mathcal{O}(x_2^3)$. Please note that the term "damping term" in this case is not correct since f_μ actually is increasing μ_t when $x_2 \to 0$ rather than damping it. However, it is common to call all low-Re number functions for "damping functions". $\boxed{P^k = \mathcal{O}(x_2^3)}$

$$\underbrace{\rho\bar{v}_1\frac{\partial\varepsilon}{\partial x_1}}_{\mathcal{O}(x_2^1)} + \underbrace{\rho\bar{v}_2\frac{\partial\varepsilon}{\partial x_2}}_{\mathcal{O}(x_2^1)} = \underbrace{C_{\varepsilon 1}\frac{\varepsilon}{k}P^k}_{\mathcal{O}(x_2^1)} + \underbrace{\frac{\partial}{\partial x_2}\left(\frac{\mu_t}{\sigma_\varepsilon}\frac{\partial\varepsilon}{\partial x_2}\right)}_{\mathcal{O}(x_2^2)} + \underbrace{\mu\frac{\partial^2\varepsilon}{\partial x_2^2}}_{\mathcal{O}(x_2^0)} - \underbrace{C_{\varepsilon 2}\rho\frac{\varepsilon^2}{k}}_{\mathcal{O}(x_2^{-2})}$$

at the wall =0 $\Longrightarrow$ $$0 = \mu\frac{\partial^2\varepsilon}{\partial x_2^2} - C_{\varepsilon 2}\rho\frac{\varepsilon^2}{k}.$$

این دو عبارت هم اندازه نبوده و به همین دلیل از تابع
خاصی استفاده می شود (در کنار دیواره)

The Launder-Sharma Low-Re $k-\varepsilon$ Models

$$\frac{\partial \rho \bar{U} k}{\partial x} + \frac{\partial \rho \bar{V} k}{\partial y} = \frac{\partial}{\partial y}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] + \mu_t \left(\frac{\partial \bar{U}}{\partial y}\right)^2 - \rho\varepsilon$$

$$\frac{\partial \rho \bar{U} \tilde{\varepsilon}}{\partial x} + \frac{\partial \rho \bar{V} \tilde{\varepsilon}}{\partial y} = \frac{\partial}{\partial y}\left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \tilde{\varepsilon}}{\partial y}\right] + c_{1\varepsilon} f_1 \frac{\tilde{\varepsilon}}{k}\mu_t \left(\frac{\partial \bar{U}}{\partial y}\right)^2 - c_{\varepsilon 2} f_2 \rho \frac{\tilde{\varepsilon}^2}{k} + E$$

$$\mu_t = c_\mu f_\mu \rho \frac{k^2}{\tilde{\varepsilon}}$$

$$\varepsilon = \tilde{\varepsilon} + D$$

$$f_\mu = \exp\left(\frac{-3.4}{(1 + R_T/50)^2}\right)$$

$$f_1 = 1$$

$$f_2 = 1 - 0.3\exp\left(-R_T^2\right)$$

$$D = 2\mu\left(\frac{\partial\sqrt{k}}{\partial y}\right)^2$$

$$E = 2\mu\frac{\mu_t}{\rho}\left(\frac{\partial^2 \bar{U}}{\partial y^2}\right)^2$$

$$R_T = \frac{k^2}{\nu\tilde{\varepsilon}}$$

$$\varepsilon_{wall} = \nu\frac{\partial^2 k}{\partial y^2}$$

$$\varepsilon_{wall} = 2\nu\left(\frac{\partial\sqrt{k}}{\partial y}\right)^2.$$

Jones-Launder Model

$$f_\mu = e^{-2.5/(1+Re_T/50)}$$
$$f_1 = 1$$
$$f_2 = 1 - 0.3e^{-Re_T^2}$$
$$\epsilon_o = 2\nu \left(\frac{\partial\sqrt{k}}{\partial y}\right)^2$$
$$E = 2\nu\nu_T \left(\frac{\partial^2 U}{\partial y^2}\right)^2$$
$$C_{\epsilon 1} = 1.45, \quad C_{\epsilon 2} = 2.00, \quad C_\mu = 0.09, \quad \sigma_k = 1.0, \quad \sigma_\epsilon = 1.3$$

$$k = \tilde{\epsilon} = 0 \quad \text{at} \quad y = 0$$

Lam-Bremhorst Model

$$f_\mu = \left(1 - e^{-0.0165 R_y}\right)^2 \left(1 + 20.5/Re_T\right)$$
$$f_1 = 1 + (0.05/f_\mu)^3$$
$$f_2 = 1 - e^{-Re_T^2}$$
$$\epsilon_o = 0$$
$$E = 0$$
$$C_{\epsilon 1} = 1.44, \quad C_{\epsilon 2} = 1.92, \quad C_\mu = 0.09, \quad \sigma_k = 1.0, \quad \sigma_\epsilon = 1.3$$

$$\epsilon = \nu \frac{\partial^2 k}{\partial y^2} \quad \text{at} \quad y = 0$$

$$\frac{\partial \epsilon}{\partial y} = 0 \quad \text{at} \quad y = 0$$

Chien Model

$$f_\mu = 1 - e^{-0.0115y^+}$$
$$f_1 = 1$$
$$f_2 = 1 - 0.22e^{-(Re_T/6)^2}$$
$$\epsilon_o = 2\nu \frac{k}{y^2}$$
$$E = -2\nu \frac{\tilde{\epsilon}}{y^2} e^{-y^+/2}$$
$$C_{\epsilon 1} = 1.35, \quad C_{\epsilon 2} = 1.80, \quad C_\mu = 0.09, \quad \sigma_k = 1.0, \quad \sigma_\epsilon = 1.3$$

$$k = \tilde{\epsilon} = 0 \quad \text{at} \quad y = 0$$

all four models guarantee

$$k \sim y^2 \quad \text{and} \quad \epsilon/k \to 2\nu/y^2 \quad \text{as} \quad y \to 0$$

The low-Re $k-\omega$ Model of Peng *et al.*

$$\frac{\partial k}{\partial t}+\frac{\partial}{\partial x_j}(\bar{U}_j k)=\frac{\partial}{\partial x_j}\left[\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]+P_k-c_k f_k \omega k \qquad \nu_t=f_\mu\frac{k}{\omega}$$

$$\frac{\partial \omega}{\partial t}+\frac{\partial}{\partial x_j}(\bar{U}_j \omega)=\frac{\partial}{\partial x_j}\left[\left(\nu+\frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right]+\frac{\omega}{k}\left(c_{\omega 1} f_\omega P_k-c_{\omega 2} k\omega\right)+c_\omega\frac{\nu_t}{k}\left(\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}\right)$$

$$f_k=1-0.722\exp\left[-\left(\frac{R_t}{10}\right)^4\right]$$

$$f_\mu=0.025+\left\{1-\exp\left[-\left(\frac{R_t}{10}\right)^{3/4}\right]\right\}\left\{0.975+\frac{0.001}{R_t}\exp\left[-\left(\frac{R_t}{200}\right)^2\right]\right\}$$

$$f_\omega=1+4.3\exp\left[-\left(\frac{R_t}{1.5}\right)^{1/2}\right],\quad f_\omega=1+4.3\exp\left[-\left(\frac{R_t}{1.5}\right)^{1/2}\right]$$

$$c_k=0.09,\ c_{\omega 1}=0.42,\ c_{\omega 2}=0.075$$
$$c_\omega=0.75,\sigma_k=0.8,\ \sigma_\omega=1.35$$

$$R_t=k/(\omega\nu)$$

turbulent Reynolds number

The ω equation is normally not solved close to the wall

The low-Re $k-\omega$ Model of Bredberg *et al.*

$$\frac{\partial k}{\partial t}+\frac{\partial}{\partial x_j}(\bar{U}_j k)=P_k-C_k k\omega+\frac{\partial}{\partial x_j}\left[\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]$$

$$\frac{\partial \omega}{\partial t}+\frac{\partial}{\partial x_j}(\bar{U}_j \omega)=C_{\omega 1}\frac{\omega}{k}P_k-C_{\omega 2}\omega^2+$$

$$C_\omega\left(\frac{\nu}{k}+\frac{\nu_t}{k}\right)\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}+\frac{\partial}{\partial x_j}\left[\left(\nu+\frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right]$$

$$\nu_t=C_\mu f_\mu \frac{k}{\omega}$$

$$f_\mu=0.09+\left(0.91+\frac{1}{R_t^3}\right)\left[1-\exp\left\{-\left(\frac{R_t}{25}\right)^{2.75}\right\}\right]$$

$$R_t=k/(\omega\nu)$$

turbulent Reynolds number

$$C_k=0.09,\quad C_\mu=1,\quad C_\omega=1.1,\quad C_{\omega 1}=0.49,$$
$$C_{\omega 2}=0.072,\quad \sigma_k=1,\quad \sigma_\omega=1.8$$

- Advantages with $k - \varepsilon$ models (or eddy viscosity models):
  - *i)* simple due to the use of an isotropic eddy (turbulent) viscosity
  - *ii)* stable via stability-promoting second-order gradients in the mean-flow equations
  - *iii)* work reasonably well for a large number of engineering flows
- Disadvantages:
  - *i)* isotropic, and thus not good in predicting normal stresses ($\overline{v_1'^2}, \overline{v_2'^2}, \overline{v_3'^2}$)
  - *ii)* as a consequence of *i)* it is unable to account for curvature effects
  - *iii)* as a consequence of *i)* it is unable to account for irrotational strains (stagnation flow)
  - *iv)* in boundary layers approaching separation, the production due to normal stresses is of the same magnitude as that due to shear stresses