

Prediction of Turbulent flow – Part 5

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Filters

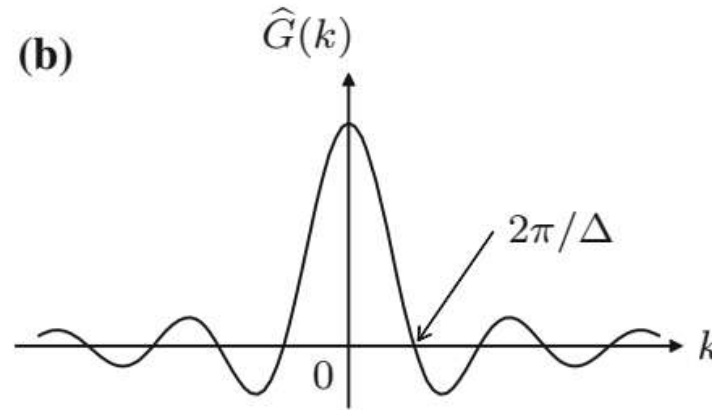
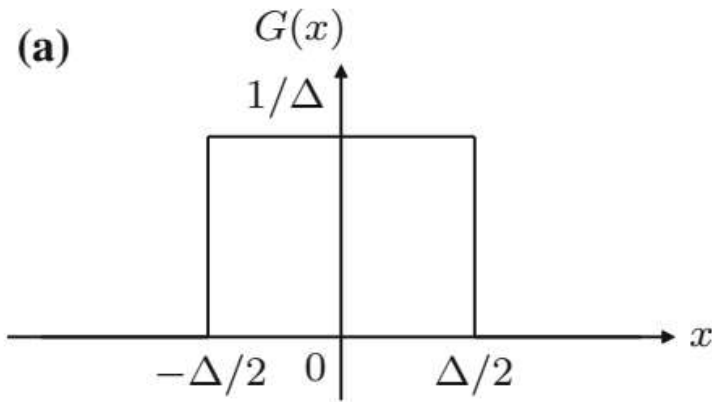
Filtering can be performed through an application of convolution

$$\bar{f}(x) = \int_{-\infty}^{\infty} G(y) f(x - y) dy,$$

where the function $G(y)$ should be positive around $y = 0$ and satisfy $\lim_{y \rightarrow \pm\infty} G(y) = 0$. We construct this function to have the property of

$$\int_{-\infty}^{\infty} G(y) dy = 1$$

so that $\bar{f}(x)$ is the weighted average of $f(x)$ near x . The function G is referred to as the *filter function*.



a The box filter and its **b** Fourier transform

$$G(x) = \begin{cases} 1/\Delta & (|x| < \Delta/2) \\ 0 & (|x| > \Delta/2) \end{cases}$$

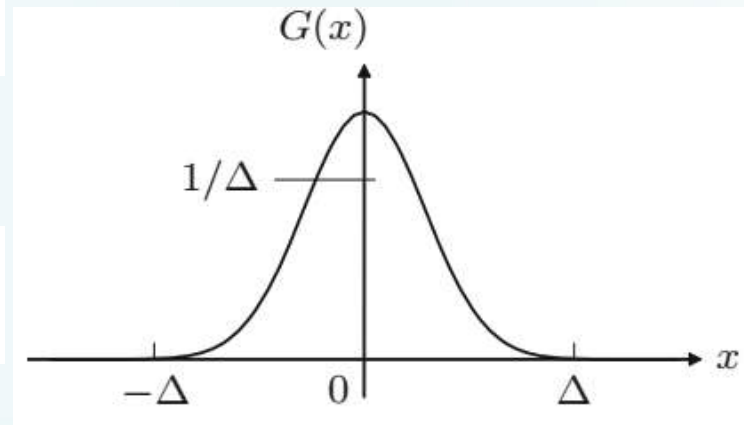
$$\hat{G}(k) = \frac{\sin(\Delta k/2)}{\Delta k/2}$$

a box filter in the wave space

$$\hat{G}(k) = \begin{cases} 1 & (|k| < \pi/\Delta) \\ 0 & (|k| > \pi/\Delta) \end{cases}$$

in physical space

$$G(x) = 2 \frac{\sin(\pi x/\Delta)}{\pi x}$$



Gaussian filter

$$G(x) = \sqrt{\frac{6}{\pi \Delta^2}} \exp\left(-\frac{6x^2}{\Delta^2}\right)$$

The Fourier transformed filter in wave space

$$\hat{G}(k) = \exp\left(-\frac{\Delta^2 k^2}{24}\right)$$

Time averaging and filtering

$$\langle \Phi \rangle = \frac{1}{2T} \int_{-T}^T \Phi(t) dt, \quad \Phi = \langle \Phi \rangle + \Phi'$$

$$\bar{\Phi}(x, t) = \frac{1}{\Delta x} \int_{x-0.5\Delta x}^{x+0.5\Delta x} \Phi(\xi, t) d\xi$$

$$\Phi = \bar{\Phi} + \Phi''$$

The equations for the filtered variables have the same form as Navier-Stokes,

$$\frac{\partial \bar{v}_i}{\partial t} + \frac{\partial}{\partial x_j} (\bar{v}_i \bar{v}_j) = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{v}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\frac{\partial \bar{v}_i}{\partial x_i} = 0$$

where the subgrid stresses are given by

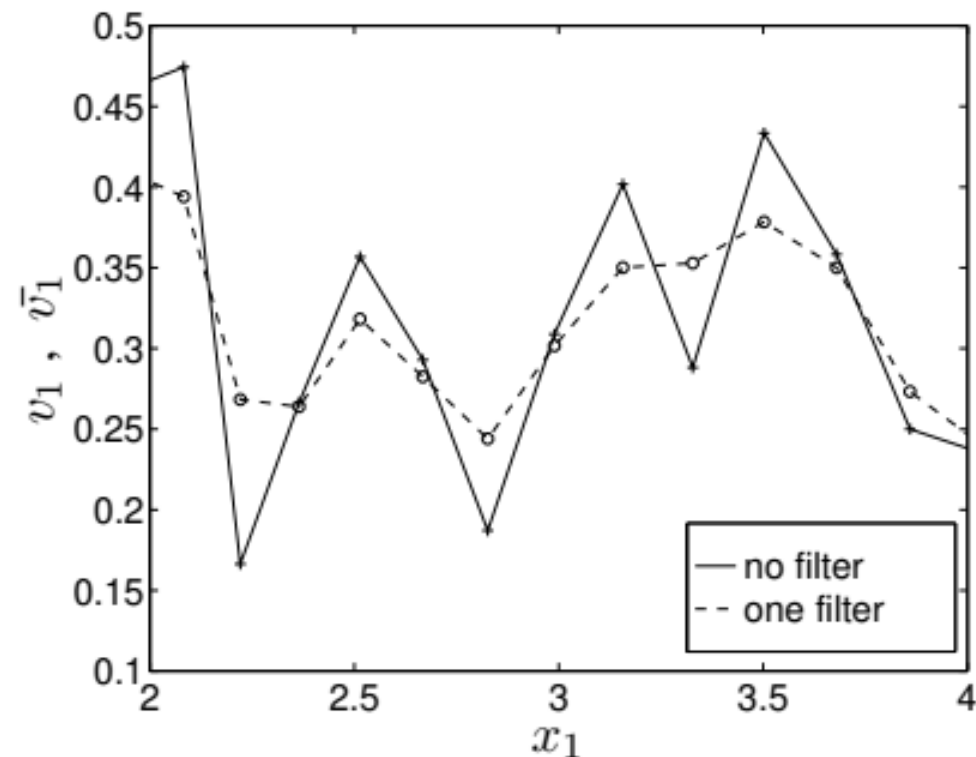
$$\tau_{ij} = \overline{v_i v_j} - \bar{v}_i \bar{v}_j$$

Contrary to Reynolds time averaging where $\langle v'_i \rangle = 0$

$$\overline{v_i''} \neq 0 \quad \bar{\bar{v}}_i \neq \bar{v}_i$$

This is true for box filters.

Note that for the spectral cut-off filter $\bar{v}_i = \bar{\bar{v}}_i$.



The box-filter and the cut-off filter

The box filtering is formally defined as (1D)

$$\bar{v}(x) = \int_{-\infty}^{\infty} G_B(r) v(x-r) dr$$
$$G_B(r) = \begin{cases} 1/\Delta, & \text{if } r \leq \Delta/2 \\ 0, & \text{if } r > \Delta/2 \end{cases}$$
$$\int_{-\infty}^{\infty} G_B(r) dr = 1$$

However, it is often convenient to study the filtering process in the spectral space.

we simply set the contribution from wavenumbers larger than cut-off to zero.

$$\hat{G}_C(\kappa) = \begin{cases} 1 & \text{if } \kappa \leq \kappa_c \\ 0 & \text{otherwise} \end{cases}$$

the cut-off wavenumber $\kappa_c = \pi/\Delta$

Let's look at the filtering of Navier-Stokes in more detail.

The pressure gradient term, for example, reads

$$\overline{\frac{\partial p}{\partial x_i}} = \frac{1}{V} \int_V \frac{\partial p}{\partial x_i} dV$$

سوال مهم این است که می شود مشتق را از انتگرال خارج کرد؟ در صورت ثابت بودن اندازه حجم کنترل دیگر تابع X نبوده و جابجایی امکان پذیر است. اما در اکثر مسایل حجم کنترل (شبکه) کاملاً غیر یکنواخت بوده و در میدان تغییر می کند.

$$\overline{\frac{\partial p}{\partial x_i}} = \frac{\partial}{\partial x_i} \left(\frac{1}{V} \int_V p dV \right) + \mathcal{O}((\Delta x)^2) = \frac{\partial \bar{p}}{\partial x_i} + \mathcal{O}((\Delta x)^2)$$

if V is a function of x_i , the error we do when moving the derivative out of the integral is proportional to $(\Delta x)^2$

جهت مطالعه به مقاله ذیل مراجعه گردد

S. Ghosal, P. Moin, [The Basic Equations for the Large Eddy Simulation of Turbulent Flows in Complex Geometry](#) [Journal of Computational Physics](#), Volume 118, Issue 1, April 1995, Pages 24-37

the convective term.

$$\overline{\frac{\partial v_i v_j}{\partial x_j}} = \frac{\partial}{\partial x_j} \left(\frac{1}{V} \int_V v_i v_j dV \right) + \mathcal{O}((\Delta x)^2) = \frac{\partial}{\partial x_j} \underbrace{(\overline{v_i v_j})}_{?} + \mathcal{O}((\Delta x)^2)$$

Differences between time-averaging (RANS) and space filtering (LES)

In RANS: $\langle \langle v \rangle \rangle = \frac{1}{2T} \int_{-T}^T \langle v \rangle dt = \frac{1}{2T} \langle v \rangle 2T = \langle v \rangle$

In LES, $\overline{\overline{v}} \neq \bar{v}$ (and since $v = \bar{v} + v''$ we get $\overline{v''} \neq 0$).

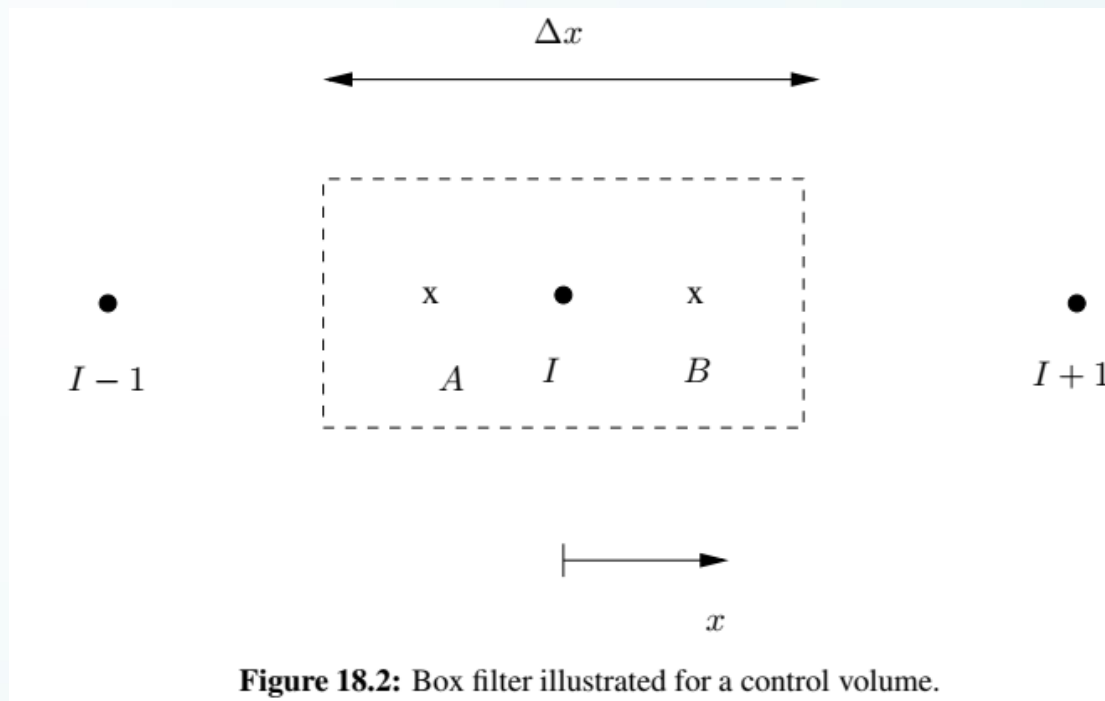


Figure 18.2: Box filter illustrated for a control volume.

Let's filter $\bar{v}_I$ once more (filter size Δx , see Fig. 18.2. For simplicity we do it in 1D. (Note that subscript I denotes node number.)

$$\begin{aligned}\bar{\bar{v}}_I &= \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \bar{v}(\xi) d\xi = \frac{1}{\Delta x} \left(\int_{-\Delta x/2}^0 \bar{v}(\xi) d\xi + \int_0^{\Delta x/2} \bar{v}(\xi) d\xi \right) = \\ &= \frac{1}{\Delta x} \left(\frac{\Delta x}{2} \bar{v}_A + \frac{\Delta x}{2} \bar{v}_B \right).\end{aligned}$$

The trapezoidal rule, which is second-order accurate, was used to estimate the integrals.

$$\bar{\bar{v}}_I = \frac{1}{2} \left[\left(\frac{1}{4} \bar{v}_{I-1} + \frac{3}{4} \bar{v}_I \right) + \left(\frac{3}{4} \bar{v}_I + \frac{1}{4} \bar{v}_{I+1} \right) \right] = \frac{1}{8} (\bar{v}_{I-1} + 6\bar{v}_I + \bar{v}_{I+1}) \neq \bar{v}_I$$

the filtered continuity and momentum equations for incompressible flow become

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \overline{u_i u_j}}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j} (2\nu \bar{D}_{ij})$$

$\bar{D}_{ij}$ is the grid-scale rate-of-strain tensor

$$\bar{D}_{ij} = \frac{1}{2} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right)$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial (\bar{u}_i \bar{u}_j)}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j} (-\tau_{ij} + 2\nu \bar{D}_{ij})$$

subgrid-scale stress (SGS stress),

$$\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j$$

$$\tau_{ij} = L_{ij} + C_{ij} + R_{ij}$$

$$\left\{ \begin{array}{ll} L_{ij} = \overline{\bar{u}_i \bar{u}_j} - \bar{u}_i \bar{u}_j & \text{Leonard term} \\ C_{ij} = \overline{\bar{u}_i u'_j} + \overline{u'_i \bar{u}_j} & \text{Cross term} \\ R_{ij} = \overline{u'_i u'_j} & \text{SGS Reynolds stress} \end{array} \right.$$

do not satisfy invariance under Galilean transformation independently

Notice that L_{ij} and C_{ij} are not referred to as stresses

²For the turbulent stress $\tau(u, v) = \overline{uv} - \bar{u} \bar{v}$ to be Galilean invariant, $\tau(u + \alpha, v + \beta) = \tau(u, v)$ must hold for a system moving at a constant velocity of (α, β) . If we let velocity U to be the constant velocity $\bar{U}$ for $u^* = u + U$, then we have $\bar{u}^* = \bar{u} + U$ and $u^{*'} = u'$. Thus the Leonard stress L_{ij} becomes

$$L_{ij} = \overline{u_i^* u_j^*} - \bar{u}_i^* \bar{u}_j^* - (\bar{u}_i^* - \bar{u}_i) \bar{u}_j - (\bar{u}_j^* - \bar{u}_j) \bar{u}_i,$$

which is not Galilean invariant since U remains in the expression. The modified L_{ij}^m on the other hand becomes

$$L_{ij}^m = \overline{u_i^* u_j^*} - \bar{u}_i^* \bar{u}_j^*,$$

proposed by Germano

which is Galilean invariant.

$$\tau_{ij} = L_{ij}^m + C_{ij}^m + R_{ij}^m \quad \text{reformulation proposed by Garmano}$$

- *Modified Leonard stress*

$$L_{ij}^m = \overline{\overline{u_i u_j}} - \overline{\overline{u_i}} \overline{\overline{u_j}}$$

- *Modified cross stress*

$$C_{ij}^m = \overline{\overline{u_i u'_j}} + \overline{\overline{u'_i u_j}} - (\overline{\overline{u_i}} \overline{\overline{u'_j}} + \overline{\overline{u'_i}} \overline{\overline{u_j}})$$

- *Modified SGS Reynolds stress*

$$R_{ij}^m = \overline{\overline{u'_i u'_j}} - \overline{\overline{u'_i}} \overline{\overline{u'_j}}$$

The modified Leonard stress L_{ij}^m is determined with the filtered grid-scale velocity but the evaluation of C_{ij}^m and R_{ij}^m requires the use of models.

The difference between the modified Leonard stress L_{ij}^m and the Leonard stress L_{ij} is

$$B_{ij} = L_{ij}^m - L_{ij} = \overline{\overline{u_i u_j}} - \overline{\overline{u_i}} \overline{\overline{u_j}}$$

This term is known as the *scale-similarity term*

Subgrid model

The simplest model is the Smagorinsky model

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -\nu_{sgs} \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) = -2\nu_{sgs}\bar{s}_{ij}$$
$$\nu_{sgs} = (C_S\Delta)^2 \sqrt{2\bar{s}_{ij}\bar{s}_{ij}} \equiv (C_S\Delta)^2 |\bar{s}|$$

the filter-width is taken as the local grid size

$$\Delta = (\Delta V_{IJK})^{1/3}$$

$$\left(\Delta = (\Delta x \times \Delta y \times \Delta z)^{1/3} \right)$$

$$|\bar{s}|^2 = \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) \frac{\partial \bar{v}_i}{\partial x_j} = 2\bar{s}_{ij}(\bar{s}_{ij} + \bar{\Omega}_{ij}) = 2\bar{s}_{ij}\bar{s}_{ij}$$
$$\bar{\Omega}_{ij} = \frac{1}{2} \left(\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial \bar{v}_j}{\partial x_i} \right)$$

$\nu_{sgs} = \mathcal{UL}$ be obtained through dimensional analysis

The turbulent velocity scale, $\mathcal{U}$, is obtained as the first term in Taylor expansion, $\mathcal{L}\partial\bar{v}/\partial x$, and the length scale, $\mathcal{L}$, is taken as $C_S\Delta$ which gives

$$\nu_{sgs} = (C_S\Delta)^2 |\bar{s}|$$

the SGS turbulent fluctuations near a wall go to zero

A damping function f_μ

$$f_\mu = 1 - \exp(-x_2^+/26)$$

A more convenient way to dampen the SGS viscosity near the wall is simply to use the RANS length scale as an upper limit, i.e.

where n is the distance to the nearest wall.

$$\Delta = \min \left\{ (\Delta V_{IJK})^{1/3}, \kappa n \right\}$$

Disadvantage of Smagorinsky model: the “constant” C_S is not constant, but it is flow-dependent. It is found to vary in the range from $C_S = 0.065$ [81] to $C_S = 0.25$ [82].

[81] P. Moin and J. Kim. Numerical investigation of turbulent channel flow. *Journal of Fluid Mechanics*, 118:341–377, 1982.

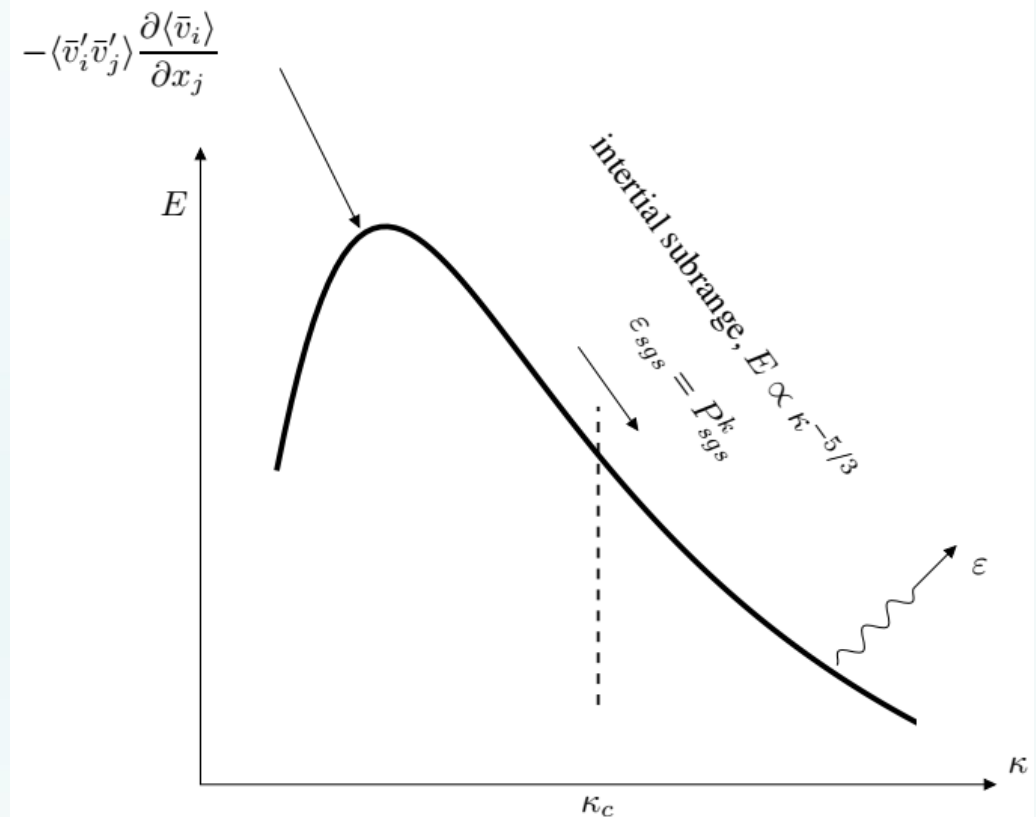
[82] W. P. Jones and M. Wille. Large eddy simulation of a jet in a cross-flow. In *10th Symp. on Turbulent Shear Flows*, pages 4:1 – 4:6, The Pennsylvania State University, 1995.

Energy path

At cut-off, SGS kinetic energy is dissipated

$$\varepsilon_{sgs} = -\tau_{ij} \bar{s}_{ij} = 2\nu_{sgs} \bar{s}_{ij} \bar{s}_{ij}$$

We assume that ALL dissipation takes place in the dissipation range. This is a good approximation, but in reality dissipation (i.e. transfer of energy from kinetic energy to internal energy which corresponds to an increase in temperature) takes place at all wave numbers, and the dissipation increases for increasing wave number



SGS kinetic energy

The SGS kinetic energy k_{sgs} can be estimated from the Kolmogorov $-5/3$ law.

$$k = \int_0^{\infty} E(\kappa) d\kappa$$

Changing the lower integration limit to wavenumbers larger than cut-off (i.e. κ_c) gives the SGS kinetic energy

$$k_{sgs} = \int_{\kappa_c}^{\infty} E(\kappa) d\kappa$$

The Kolmogorov $-5/3$ law now gives

$$k_{sgs} = \int_{\kappa_c}^{\infty} C_K \kappa^{-5/3} \varepsilon^{2/3} d\kappa$$

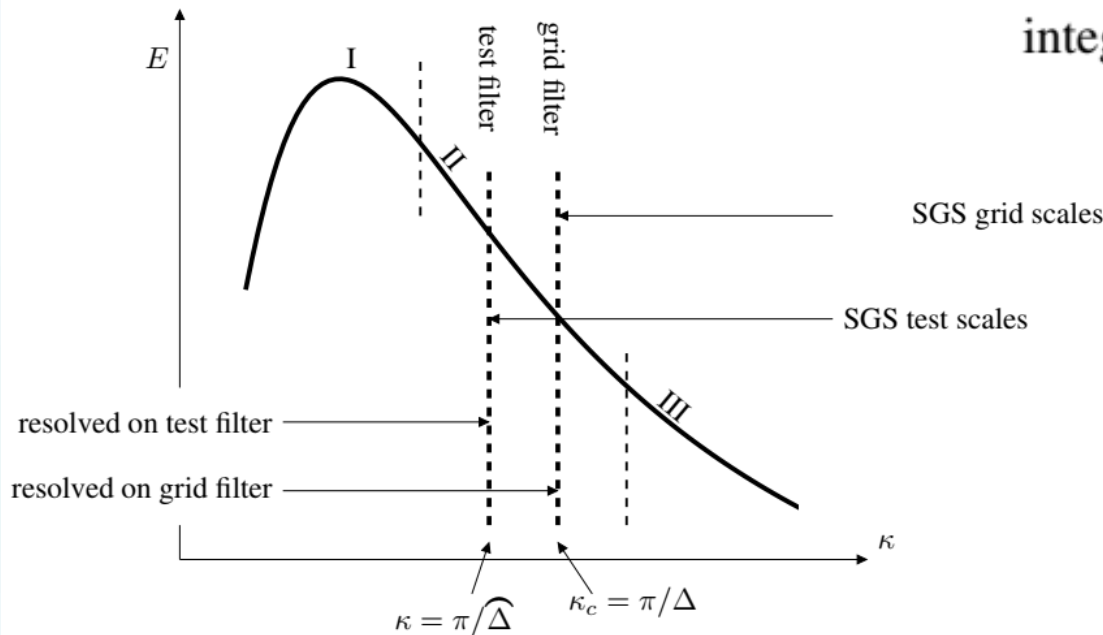
where $C_K = 1.5$

integration and replacing κ_c with π/Δ we get

$$k_{sgs} = \frac{3}{2} C_K \left(\frac{\Delta \varepsilon}{\pi} \right)^{2/3}$$

the resolved turbulent kinetic energy, k_{res} ,

$$k_{res} = \int_0^{\kappa_c} E(\kappa) d\kappa$$



LES vs. RANS

LES can handle many flows which RANS (Reynolds Averaged Navier Stokes) cannot; the reason is that in LES large, turbulent scales are resolved. Examples are:

- o Flows with large separation
 - o Bluff-body flows (e.g. flow around a car); the wake often includes large, unsteady, turbulent structures
 - o Transition
- In RANS all turbulent scales are modeled $\Rightarrow$ inaccurate
 - In LES only small, isotropic turbulent scales are modeled $\Rightarrow$ accurate
 - LES is *very* much more expensive than RANS.

The dynamic model

In this model the constant C is not arbitrarily chosen (or optimized), but it is computed.
apply two filters to Navier-Stokes

[grid filter and a second, coarser filter (test filter, denoted by $\frown$)] where $\widehat{\Delta} = 2\Delta$ we get

$$\frac{\partial \widehat{\bar{v}}_i}{\partial t} + \frac{\partial}{\partial x_j} \left(\widehat{\bar{v}}_i \widehat{\bar{v}}_j \right) = -\frac{1}{\rho} \frac{\partial \widehat{\bar{p}}}{\partial x_i} + \nu \frac{\partial^2 \widehat{\bar{v}}_i}{\partial x_j \partial x_j} - \frac{\partial T_{ij}}{\partial x_j}$$

$$\frac{\partial \widehat{v}_i}{\partial t} + \frac{\partial}{\partial x_j} \left(\widehat{v}_i \widehat{v}_j \right) = -\frac{1}{\rho} \frac{\partial \widehat{p}}{\partial x_i} + \nu \frac{\partial^2 \widehat{v}_i}{\partial x_j \partial x_j} - \frac{\partial T_{ij}}{\partial x_j}$$

$$T_{ij} = \overline{v_i v_j} - \widehat{v}_i \widehat{v}_j$$

$$\frac{\partial \widehat{v}_i}{\partial t} + \frac{\partial}{\partial x_j} \left(\widehat{v}_i \widehat{v}_j \right) = -\frac{1}{\rho} \frac{\partial \widehat{p}}{\partial x_i} + \nu \frac{\partial^2 \widehat{v}_i}{\partial x_j \partial x_j} - \frac{\partial \widehat{\tau}_{ij}}{\partial x_j} - \frac{\partial}{\partial x_j} \left(\overline{v_i v_j} - \widehat{v}_i \widehat{v}_j \right)$$

$$\overline{v_i v_j} - \widehat{v}_i \widehat{v}_j + \widehat{\tau}_{ij} = T_{ij}$$

The *dynamic* Leonard stresses are now defined as

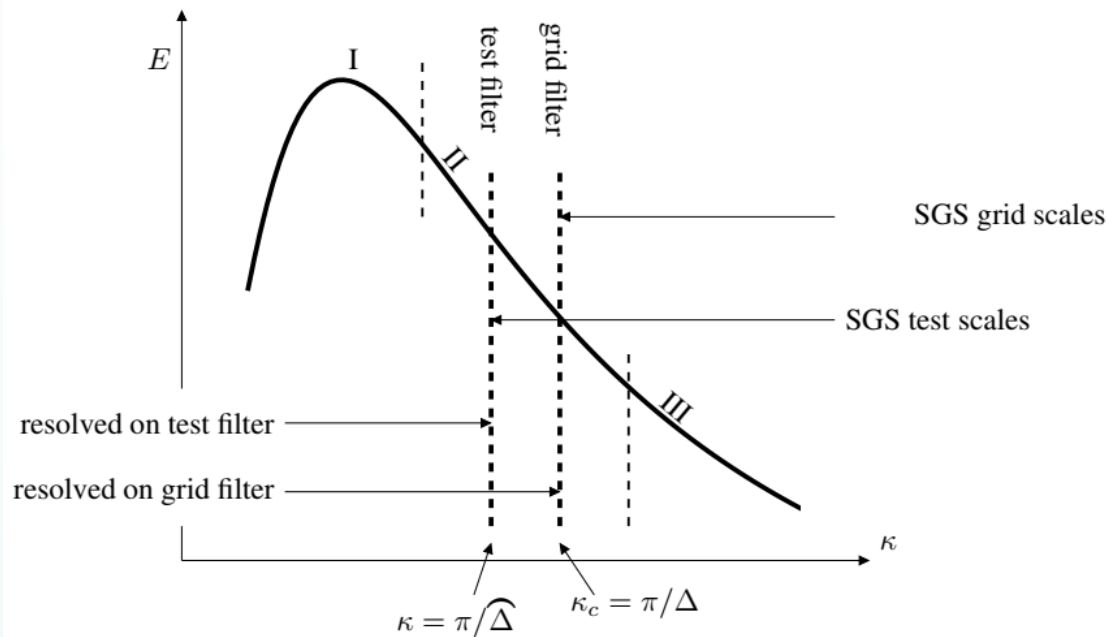
$$\mathcal{L}_{ij} \equiv \overline{v_i v_j} - \widehat{v}_i \widehat{v}_j = T_{ij} - \widehat{\tau}_{ij}$$

$$\mathcal{L}_{ii} \equiv T_{ii} - \widehat{\tau}_{ii}$$

With this expression we can re-formulate

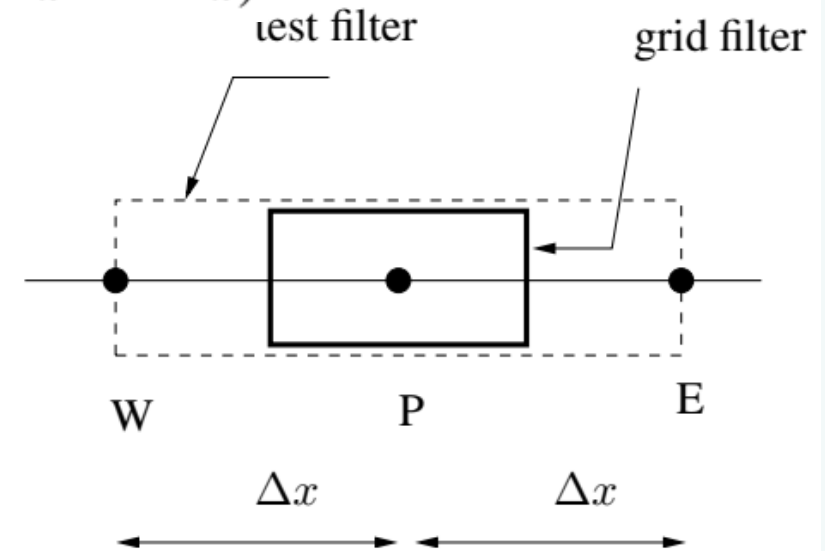
$$\mathcal{L}_{ij} - \frac{1}{3} \delta_{ij} \mathcal{L}_{kk} = T_{ij} - \frac{1}{3} \delta_{ij} T_{kk} - \left(\widehat{\tau}_{ij} - \frac{1}{3} \delta_{ij} \widehat{\tau}_{kk} \right)$$

In the energy spectrum, the test filter is located at lower wave number than the grid filter



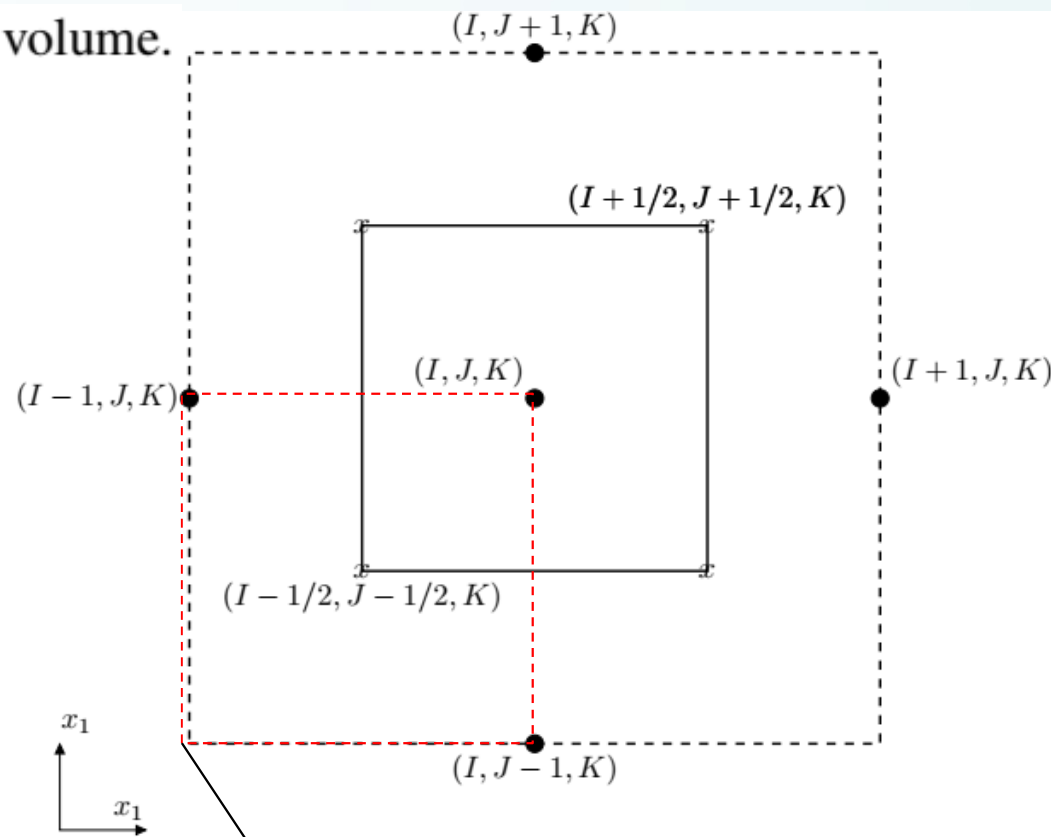
The test filter

($\widehat{\Delta x} = 2\Delta x$) the 1D example



$$\begin{aligned}\widehat{\bar{v}}_P &= \frac{1}{2\Delta x} \int_W^E \bar{v} dx = \frac{1}{2\Delta x} \left(\int_W^P \bar{v} dx + \int_P^E \bar{v} dx \right) \\ &= \frac{1}{2\Delta x} (\bar{v}_w \Delta x + \bar{v}_e \Delta x) = \frac{1}{2} \left(\frac{\bar{v}_W + \bar{v}_P}{2} + \frac{\bar{v}_P + \bar{v}_E}{2} \right) \\ &= \frac{1}{4} (\bar{v}_W + 2\bar{v}_P + \bar{v}_E)\end{aligned}$$

A 2D test filter control volume.



$$\bar{v}_{I-1/2, J-1/2, K}^f = \frac{1}{4}(\bar{v}_{I-1, J-1, K} + \bar{v}_{I-1, J, K} + \bar{v}_{I, J-1, K} + \bar{v}_{I, J, K})$$

the test filtered variable is computed as

$$\widehat{\bar{v}}_{I, J, K} = \frac{1}{4}(\bar{v}_{I-1/2, J-1/2, K}^f + \bar{v}_{I-1/2, J+1/2, K}^f + \bar{v}_{I+1/2, J-1/2, K}^f + \bar{v}_{I+1/2, J+1/2, K}^f)$$

3D filtering

For the bottom-lower-left cube, for example

$$\bar{v}_{I-1/2, J-1/2, K-1/2}^f = \frac{1}{8}(\bar{v}_{I-1, J-1, K} + \bar{v}_{I-1, J, K} + \bar{v}_{I, J-1, K} + \bar{v}_{I, J, K} \\ + \bar{v}_{I-1, J-1, K-1} + \bar{v}_{I-1, J, K-1} + \bar{v}_{I, J-1, K-1} + \bar{v}_{I, J, K-1})$$

the test filtered variable is computed as

$$\widehat{\bar{v}}_{I, J, K} = \frac{1}{8}(\bar{v}_{I-1/2, J-1/2, K-1/2}^f + \bar{v}_{I+1/2, J-1/2, K-1/2}^f \\ + \bar{v}_{I-1/2, J+1/2, K-1/2}^f + \bar{v}_{I+1/2, J+1/2, K-1/2}^f \\ + \bar{v}_{I-1/2, J-1/2, K+1/2}^f + \bar{v}_{I+1/2, J-1/2, K+1/2}^f \\ + \bar{v}_{I-1/2, J+1/2, K+1/2}^f + \bar{v}_{I+1/2, J+1/2, K+1/2}^f)$$

Stresses on grid, test and intermediate level

$$\tau_{ij} = \overline{v_i v_j} - \bar{v}_i \bar{v}_j \quad \text{stresses with } \ell < \Delta$$

$$T_{ij} = \overline{\widehat{v_i v_j}} - \widehat{\bar{v}}_i \widehat{\bar{v}}_j \quad \text{stresses with } \ell < \widehat{\Delta}$$

$$\mathcal{L}_{ij} = T_{ij} - \widehat{\tau}_{ij} \quad \text{stresses with } \Delta < \ell < \widehat{\Delta}$$

Thus the dynamic Leonard stresses represent the stresses with lengthscale, ℓ , in the range between Δ and $\widehat{\Delta}$.

Assume now that the same functional form for the subgrid stresses that is used at the grid level (τ_{ij}) also can be used at the test filter level (T_{ij}). If we use the Smagorinsky model we get

$$\tau_{ij} - \frac{1}{3} \delta_{ij} \tau_{kk} = -2C \Delta^2 |\bar{s}| \bar{s}_{ij}$$

$$T_{ij} - \frac{1}{3} \delta_{ij} T_{kk} = -2C \widehat{\Delta}^2 |\widehat{s}| \widehat{s}_{ij}$$

$$\widehat{s}_{ij} = \frac{1}{2} \left(\frac{\partial \widehat{\bar{v}}_i}{\partial x_j} + \frac{\partial \widehat{\bar{v}}_j}{\partial x_i} \right), \quad |\widehat{s}| = \left(2 \widehat{s}_{ij} \widehat{s}_{ij} \right)^{1/2}$$

(assuming that C varies slowly),

$$\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} = -2C \left(\widehat{\Delta}^2 |\widehat{s}| \widehat{s}_{ij} - \Delta^2 \overline{|\bar{s}| \bar{s}_{ij}} \right)$$

$$C = C(x_i, t)$$

where we used the relation

$$\frac{1}{3}\delta_{ij}\mathcal{L}_{kk} = \frac{1}{3}\delta_{ij}T_{kk} - \frac{1}{3}\delta_{ij}\widehat{\tau}_{kk}$$

Let us define the error as the difference between the left-hand side and the right-hand side

$$Q = \left(\mathcal{L}_{ij} - \frac{1}{3}\delta_{ij}\mathcal{L}_{kk} + 2CM_{ij} \right)^2$$

$$M_{ij} = \left(\widehat{\Delta}^2 |\widehat{s}| \widehat{s}_{ij} - \Delta^2 \overline{|\bar{s}| \bar{s}_{ij}} \right)$$

The error, Q , has a minimum (or maximum) when $\partial Q / \partial C = 0$.

D. K. Lilly. A proposed modification of the Germano subgrid-scale closure method. *Physics of Fluids A*, 4:633–635, 1992.

$$\frac{\partial Q}{\partial C} = 4M_{ij} (\mathcal{L}_{ij} + 2CM_{ij}) = 0 \quad (18.40)$$

Note that $\frac{1}{3}\delta_{ij}\mathcal{L}_{kk}M_{ij} = \frac{1}{3}\mathcal{L}_{kk}M_{ii} = 0$ since $\widehat{s}_{ii} = \bar{s}_{ii} = 0$ thanks to continuity. Since $\partial^2 Q / \partial C^2 = 8M_{ij}M_{ij} > 0$ we find that Eq. 18.40 represents indeed a minimum.

$$C = -\frac{\mathcal{L}_{ij}M_{ij}}{2M_{ij}M_{ij}}$$

It turns out that the dynamic coefficient C fluctuates wildly both in space and time. This causes numerical problems, and it has been found necessary to average C in homogeneous direction(s). Furthermore, C must be clipped to ensure that the total viscosity stays positive ($\nu + \nu_{sgs} \geq 0$).

In real 3D flows, there is no homogeneous direction. Usually local averaging and clipping (i.e. requiring that C stays within pre-defined limits) of the dynamic coefficient is used.

Bardina Model

$$u'_i = u_i - \bar{u}_i \quad \bar{u}'_i = \bar{u}_i - \bar{\bar{u}}_i$$

not zero.

$$\overline{u'_i \bar{u}_j} = (\bar{u}_i - \bar{\bar{u}}_i) \bar{\bar{u}}_j, \quad \overline{\bar{u}_i u'_j} = \bar{\bar{u}}_i (\bar{u}_j - \bar{\bar{u}}_j)$$

$$\overline{u'_i u'_j} = (\bar{u}_i - \bar{\bar{u}}_i) (\bar{u}_j - \bar{\bar{u}}_j).$$

$$C_{ij} + R_{ij} = \overline{u'_i \bar{u}_j} + \overline{\bar{u}_i u'_j} + \overline{u'_i u'_j} = \bar{u}_i \bar{u}_j - \bar{\bar{u}}_i \bar{\bar{u}}_j,$$

This is referred to as the *scale-similarity model*
or the *Bardina model*

$$\tau_{ij} = L_{ij} + C_{ij} + R_{ij},$$

$$L_{ij} = \overline{\bar{u}_i \bar{u}_j} - \bar{u}_i \bar{u}_j$$

$$\tau_{ij} = \overline{\bar{u}_i \bar{u}_j} - \bar{\bar{u}}_i \bar{\bar{u}}_j$$

The scale-similarity model is not restricted by
such assumption

However, the scale-similarity model is
not commonly used by itself, since it does not introduce any dissipation and can
destabilize the numerical calculation

this model was not sufficiently dissipative

The subgrid-scale components include vortices (or vortical structures) of all scales that were removed by the filtering operation. The scale-similarity model is suitable for relating the grid-scale and the relatively large-scale structures in the subgrid-scale components. The Smagorinsky model based on the use of eddy viscosity represents the unidirectional effect of dissipation and captures its average property at the smallest scale. In other words, the scale-similarity and eddy-viscosity models have different mechanisms as depicted in Fig. 8.7. Combining the two approaches becomes sensible if we consider the properties of the subgrid-scale components.

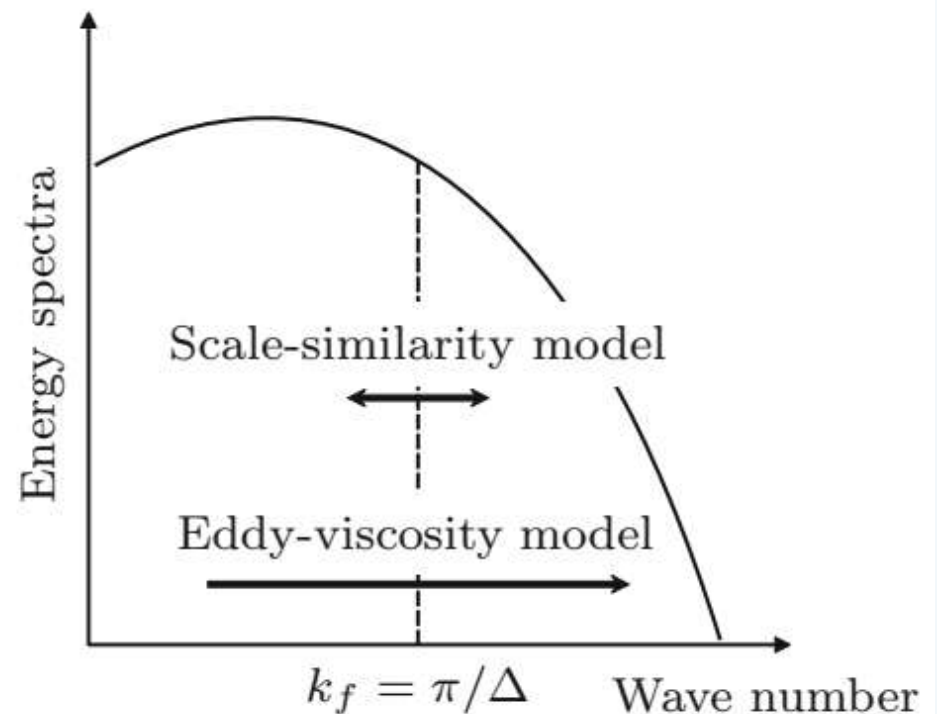
Fig. 8.7 Concept of the mixed model

$$\tau_{ij}^a = L_{ij}^{ma} - 2 (C_s \Delta)^2 |\bar{\mathbf{D}}| \bar{D}_{ij}.$$

In this model, L_{ij}^m is directly computed

$C_{ij}^m + R_{ij}^m$ is provided by the Smagorinsky model.

$$\tau_{ij} = \overline{\bar{v}_i \bar{v}_j} - \bar{\bar{v}_i} \bar{\bar{v}_j} - 2(C_S \Delta)^2 |\bar{\mathbf{s}}| \bar{s}_{ij}$$



Coherent Structure Model

Kobayashi proposed the *coherent structure model* (Kobayashi model) to determine the SGS eddy viscosity coefficient based on the turbulent structures in the flow.

$$\frac{\partial \bar{u}_i}{\partial x_j} = \bar{D}_{ij} + \bar{W}_{ij} \quad \bar{D}_{ij} = \frac{1}{2} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right), \quad \bar{W}_{ij} = \frac{1}{2} \left(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i} \right)$$

$$|\bar{\mathbf{D}}| = \sqrt{2\bar{D}_{ij}\bar{D}_{ij}}, \quad |\bar{\mathbf{W}}| = \sqrt{2\bar{W}_{ij}\bar{W}_{ij}}$$

$$\bar{Q} = \frac{|\bar{\mathbf{W}}|^2 - |\bar{\mathbf{D}}|^2}{4} = -\frac{1}{2} \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_j}{\partial x_i},$$

$$\bar{E} = \frac{|\bar{\mathbf{W}}|^2 + |\bar{\mathbf{D}}|^2}{4} = \frac{1}{2} \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_j},$$

$\bar{Q}$ is the second invariant of the grid-scale velocity gradient tensor (Q -criterion)

$\bar{E}$ is the squared magnitude of the grid-scale velocity gradient tensor.

$\bar{E}$ is proportional to the rate of kinetic energy dissipation

the *coherent structure function* $F_{CS} = \frac{\bar{Q}}{\bar{E}},$

which can be viewed as a normalized second invariant with a range of $|F_{CS}| < 1$. This function approaches 1 when the flow is under strong rotation. On the other hand, the function approaches -1 when the flow is shear-dominated compared to rotation. In the coherent structure model, the subgrid-scale stress

$$\tau_{ij}^a = -2C \Delta^2 |\overline{\mathbf{D}}| \overline{D}_{ij} \quad C = \frac{1}{20} |F_{CS}|^{\frac{3}{2}}.$$

The coherent structure function approaches zero with an asymptotic profile of $F_{CS} \propto y^2$ near a flat wall y representing the wall-normal direction

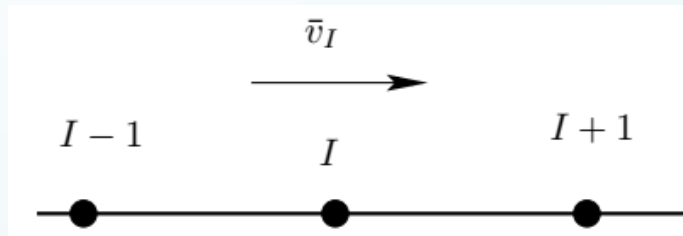
$$C = \frac{1}{22} |F_{CS}|^{\frac{3}{2}} F_{\Omega}, \quad F_{\Omega} = 1 - F_{CS}$$

The SGS eddy-viscosity coefficient for the coherent structure model

$$\nu_e = 0.05 \Delta^2 \left(\frac{|\overline{\mathbf{W}}|^2 - |\overline{\mathbf{D}}|^2}{|\overline{\mathbf{W}}|^2 + |\overline{\mathbf{D}}|^2} \right)^{\frac{3}{2}} |\overline{\mathbf{D}}|$$

Numerical dissipation

The main function of an SGS model is to dissipate (i.e. to dampen) resolved turbulent fluctuations. The SGS model is – hopefully – designed to give a proper amount of dissipation. This is the reason why in LES we should use a central differencing scheme, because this class of schemes does not give any *numerical* dissipation. All upwind schemes give numerical dissipation in addition to the modeled SGS dissipation.



assumed $\bar{v}_I > 0$. Taylor expansion gives

$$\bar{v}_{I-1} = \bar{v}_I - \Delta x \frac{\partial \bar{v}}{\partial x} + \frac{1}{2}(\Delta x)^2 \frac{\partial^2 \bar{v}}{\partial x^2} + \mathcal{O}((\Delta x)^3)$$

$$\bar{v} \left(\frac{\partial \bar{v}}{\partial x} \right)_{exact} = \bar{v}_I \left(\frac{\bar{v}_I - \bar{v}_{I-1}}{\Delta x} + \mathcal{O}(\Delta x) \right)$$

$$\frac{\bar{v}_I - \bar{v}_{I-1}}{\Delta x} = \frac{\partial \bar{v}}{\partial x} - \frac{1}{2} \Delta x \frac{\partial^2 \bar{v}}{\partial x^2} + \mathcal{O}((\Delta x)^2)$$

$$\bar{v} \left(\frac{\partial \bar{v}}{\partial x} \right)_{exact} = \bar{v} \frac{\partial \bar{v}}{\partial x} - \underbrace{\frac{1}{2} \Delta x \bar{v} \frac{\partial^2 \bar{v}}{\partial x^2}}_{\mathcal{O}(\Delta x)} + \bar{v} \mathcal{O}((\Delta x)^2)$$

the second term on the right side corresponds to the error term

When this expression is inserted into the LES momentum equations, the second term on the right-hand side will act as an additional (numerical) diffusion term. The total diffusion term will have the form

$$\text{diffusion term} = \frac{\partial}{\partial x} \left\{ (\nu + \nu_{sgs} + \nu_{num}) \frac{\partial \bar{v}}{\partial x} \right\}$$

where the additional numerical viscosity, $\nu_{num} = 0.5|\bar{v}|\Delta x$,

This means that the total dissipation due to SGS viscosity and numerical viscosity is

$$\varepsilon_{sgs+num} = 2(\nu_{sgs} + \nu_{num}) \bar{s}_{ij} \bar{s}_{ij}$$

	RANS	LES
Domain	2D or 3D	always 3D
Time domain	steady or unsteady	always unsteady
Space discretization	2nd order upwind	central differencing
Time discretization	1st order	2nd order (e.g. C-N)
Turbulence model	more than two-equations	zero- or one-equation

Table 18.1: Differences between a finite volume RANS and LES code.

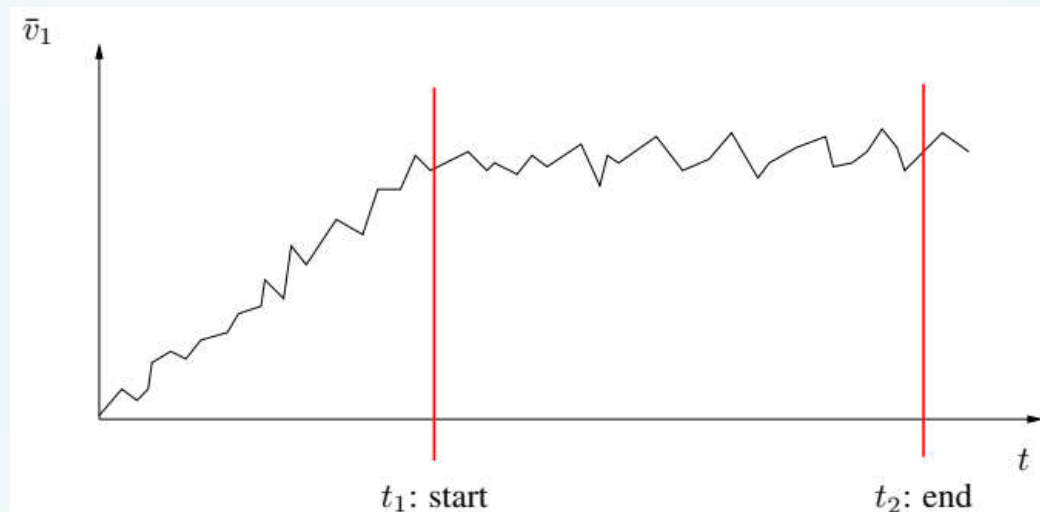


Figure 18.12: Time averaging in LES.

Both in LES and the windtunnel, the recorded time history of the $\bar{v}_1$ velocity at a point may look like in Fig. 18.12. Time averaging can start at time t_1 when the flow seems to have reached fully developed conditions. It is difficult to judge for how long one should carry out time averaging. Usually it is a good idea to form a characteristic time scale from a velocity, V (free-stream or bulk velocity), and a length scale, L (width of a wake or a body, length of a recirculation region), and use this to estimate the required averaging time; 100 time units, i.e. $100L/V$, may be a suitable averaging time for the flow around a bluff body; a value of 10 may be sufficient if L is the length of a recirculation region.

One-equation k_{sgs} model

A one-equation model can be used to model the SGS turbulent kinetic energy. The equation can be written on the same form as the RANS k -equation, i.e.

$$\frac{\partial k_{sgs}}{\partial t} + \frac{\partial}{\partial x_j}(\bar{v}_j k_{sgs}) = \frac{\partial}{\partial x_j} \left[(\nu + \nu_{sgs}) \frac{\partial k_{sgs}}{\partial x_j} \right] + P_{k_{sgs}} - \varepsilon$$

$$\nu_{sgs} = c_k \Delta k_{sgs}^{1/2}, \quad P_{k_{sgs}} = 2\nu_{sgs} \bar{s}_{ij} \bar{s}_{ij}, \quad \varepsilon = C_\varepsilon \frac{k_{sgs}^{3/2}}{\Delta}$$

Resolution requirements

The near-wall grid spacing should be about one wall unit in the wall-normal direction.

$$x_1^+ \lesssim 100(\text{streamwise}) \text{ and } x_3^+ \lesssim 30(\text{spanwise}) \text{ and, of course, } x_2^+ < 1.$$

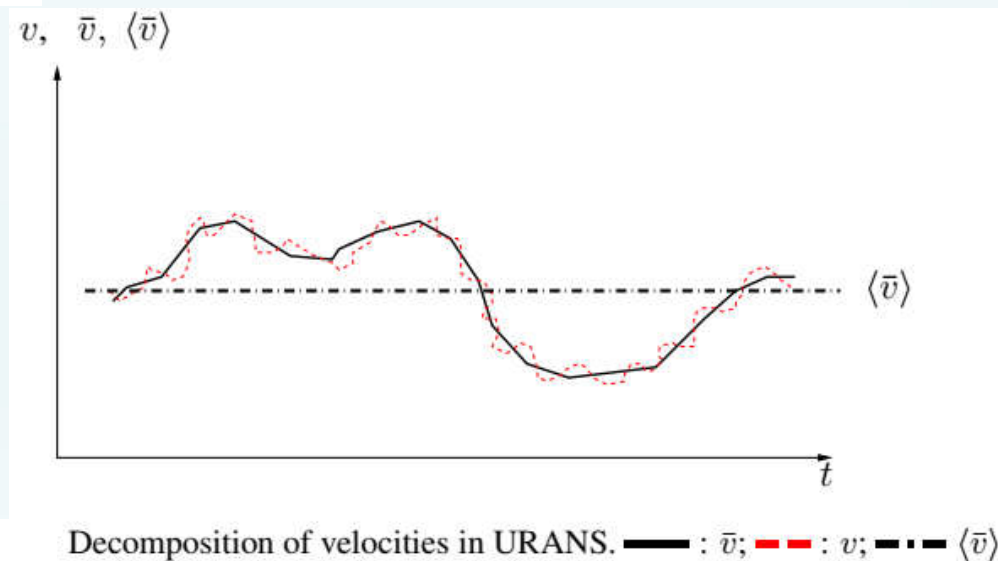
the requirement of near-wall grid resolution is the main reason why LES is too expensive for engineering flows,

An alternative to LES for industrial flows can be **unsteady** RANS (Reynolds-Averaged Navier-Stokes), often denoted **URANS** (Unsteady RANS). In URANS the usual Reynolds decomposition is employed, i.e.

$$\bar{v}(t) = \frac{1}{2T} \int_{t-T}^{t+T} v(t) dt, \quad v = \bar{v} + v''$$

$$v = \bar{v} + v'' = \langle \bar{v} \rangle + \bar{v}' + v''$$

a time-averaged part, $\langle \bar{v} \rangle$ resolved fluctuation, $\bar{v}'$, and the modeled, turbulent fluctuation, v''



Discretization

In LES it is well-known that non-dissipative discretization schemes should be used. The reason is that we don't want to dampen out resolved, turbulent fluctuations.

This is to some extent true also for URANS

with fully implicit first-order discretization in time; this gives first-order accuracy in both space and time. The turbulence model that was used was the standard $k - \varepsilon$ model. Thus, both the discretization and the turbulence model have high dissipation. The reason why the unsteadiness in these computations was not dampened out is that the vortex shedding in this flow is very strong.

In general, a discretization scheme which has little numerical dissipation should be used. How dissipative a scheme needs to be in order to be stable is flow dependent; for some simple flows, it may work with no dissipation at all (i.e. central differencing), whereas for industrially complex flows maybe a bounded second-order scheme must be used. For time discretization, the second-order accurate Crank-Nicolson works in most cases.