بسم الله الرحمن الرحيم

Convective Heat Transfer

References:

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2-Convective Heat and Mass Transfer, S. Mostafa Ghiaasiaan, 2011

3-Convective Heat Transfer, Ioan I. Pop, Derek B. Ingham, 2001

4-Heat Convection, Latif M. Jiji, 2006

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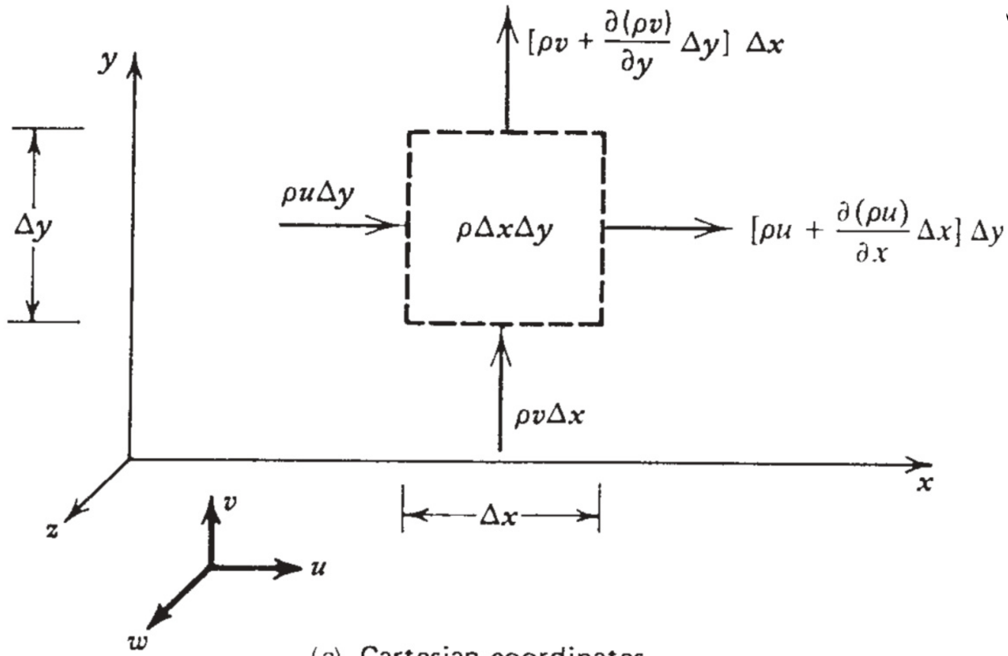
MASS CONSERVATION

From engineering thermodynamics

$$\frac{\partial M_{cv}}{\partial t} = \sum_{\substack{\text{inlet}\\ \text{ports}}} \dot{m} - \sum_{\substack{\text{outlet}\\ \text{ports}}} \dot{m}$$

$$\frac{\partial}{\partial t}(\rho\ \Delta x\ \Delta y) = \rho u\ \Delta y + \rho v\ \Delta x - \left[\rho u + \frac{\partial\ (\rho u)}{\partial x}\Delta x\right]\Delta y - \left[\rho v + \frac{\partial\ (\rho v)}{\partial y}\Delta y\right]\Delta x$$



(a) Cartesian coordinates

dividing through by the constant size of the control volume ($\Delta x\ \Delta y$)

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

In a three-dimensional flow

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \Rightarrow \frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0$$

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$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{v} = 0$$

$\mathbf{v}$ is the velocity vector (u, v, w)

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

For incompressible fluid ➡ $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

in cylindrical coordinates

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

spherical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}(v_\phi \sin\phi) + \frac{1}{\sin\phi}\frac{\partial v_\theta}{\partial \theta} = 0$$

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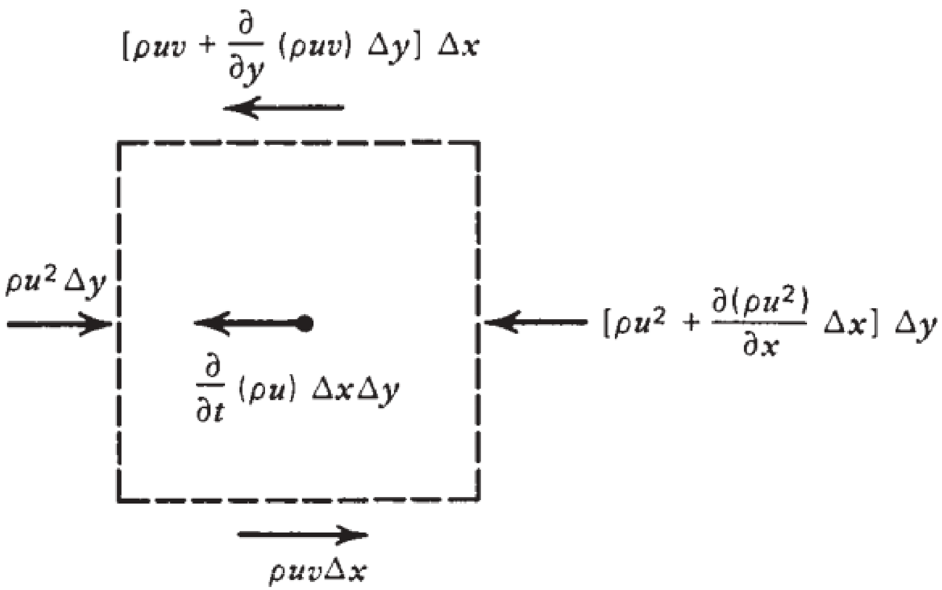
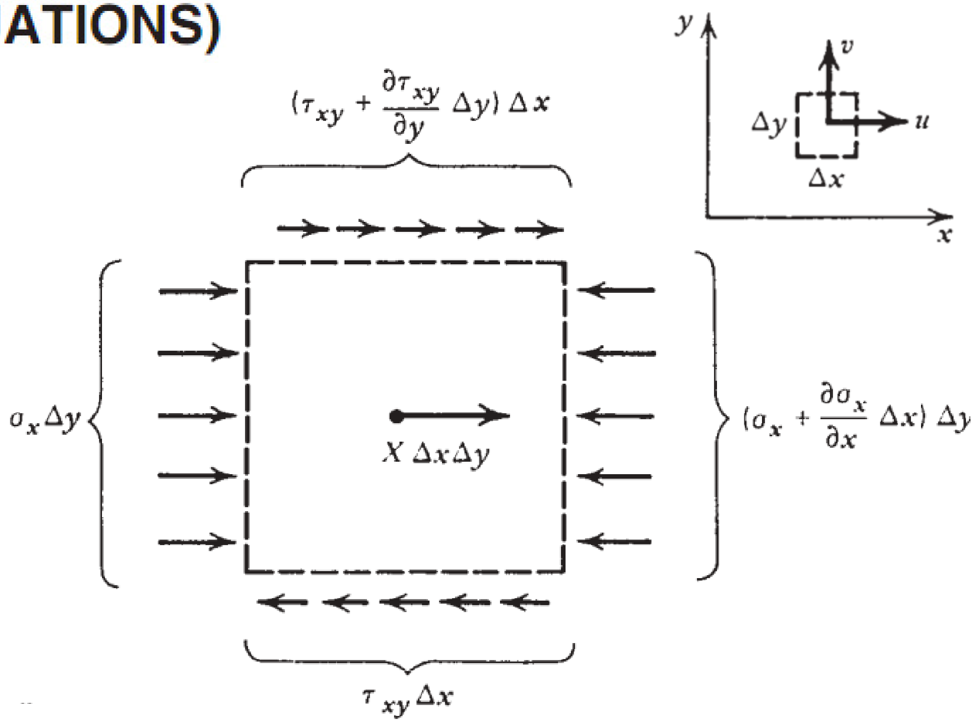
FORCE BALANCES (MOMENTUM EQUATIONS)





$$\frac{\partial}{\partial t}(Mv_n)_{\text{cv}} = \sum F_n + \sum_{\substack{\text{inlet}\\\text{ports}}} \dot{m}v_n - \sum_{\substack{\text{outlet}\\\text{ports}}} \dot{m}v_n$$

x-direction

$$\begin{aligned}
&-\frac{\partial}{\partial t}(\rho u\,\Delta x\,\Delta y) + \rho u^2\,\Delta y - \left[\rho u^2 + \frac{\partial}{\partial x}(\rho u^2)\,\Delta x\right]\Delta y \\
&+ \rho uv\,\Delta x - \left[\rho uv + \frac{\partial}{\partial y}(\rho uv)\,\Delta y\right]\Delta x \\
&+ \sigma_x\,\Delta y - \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x\right)\Delta y - \tau_{xy}\,\Delta x \\
&+ \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}\Delta y\right)\Delta x + X\,\Delta x\,\Delta y = 0
\end{aligned}$$

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dividing by $\Delta x\ \Delta y$ in the limit $(\Delta x, \Delta y) \rightarrow 0$

$$\rho \frac{Du}{Dt} + u\left[\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] = -\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X$$

$= 0$

$$\rho \frac{Du}{Dt} = -\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X$$

$$\sigma_x = P - 2\mu \frac{\partial u}{\partial x} + \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

the *Navier—Stokes equation*

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left[2\mu \frac{\partial u}{\partial x} - \frac{2\mu}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + X$$

incompressible →

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + X$$

three-dimensional flow in the (x, y, z), (u, v, w) Cartesian system

$(\rho, \mu) \cong$ constant

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+X$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+Y$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+Z$$

$$\rho\frac{D\mathbf{v}}{Dt}=-\nabla P+\mu\,\nabla^2\mathbf{v}+\mathbf{F}$$

F is the body force vector per unit volume (X, Y, Z)

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Cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)$$

$$=-\frac{\partial P}{\partial r}+\mu\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+F_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+F_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$

$$=-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+F_z$$

Spherical coordinates

$$\rho\left(\frac{Dv_r}{Dt}-\frac{v_\phi^2+v_\theta^2}{r}\right)=-\frac{\partial P}{\partial r}+\mu\left(\nabla^2 v_r-\frac{2v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi}-\frac{2v_\phi\cot\phi}{r^2}-\frac{2}{r^2\sin\phi}\frac{\partial v_\theta}{\partial\theta}\right)+F_r$$

$$\rho\left(\frac{Dv_\phi}{Dt}+\frac{v_r v_\phi}{r}-\frac{v_\theta^2\cot\phi}{r}\right)=-\frac{1}{r}\frac{\partial P}{\partial \phi}+\mu\left(\nabla^2 v_\phi+\frac{2}{r^2}\frac{\partial v_r}{\partial \phi}-\frac{v_\phi}{r^2\sin^2\phi}-\frac{2\cos\phi}{r^2\sin^2\phi}\frac{\partial v_\theta}{\partial\theta}\right)+F_\phi$$

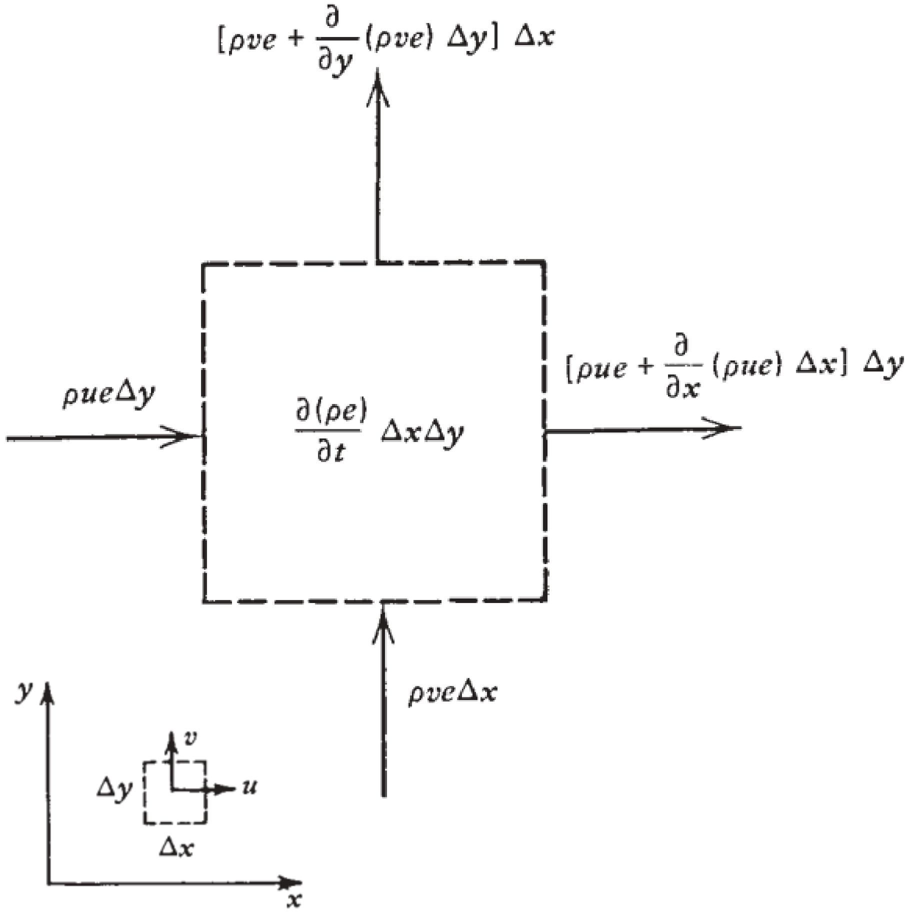
$$\rho\left(\frac{Dv_\theta}{Dt}+\frac{v_\theta v_r}{r}+\frac{v_\phi v_\theta\cot\phi}{r}\right)=-\frac{1}{r\sin\phi}\frac{\partial P}{\partial \theta}+\mu\left(\nabla^2 v_\theta-\frac{v_\theta}{r^2\sin^2\phi}+\frac{2}{r^2\sin\phi}\frac{\partial v_r}{\partial\theta}+\frac{2\cos\phi}{r^2\sin^2\phi}\frac{\partial v_\phi}{\partial\theta}\right)+F_\theta$$

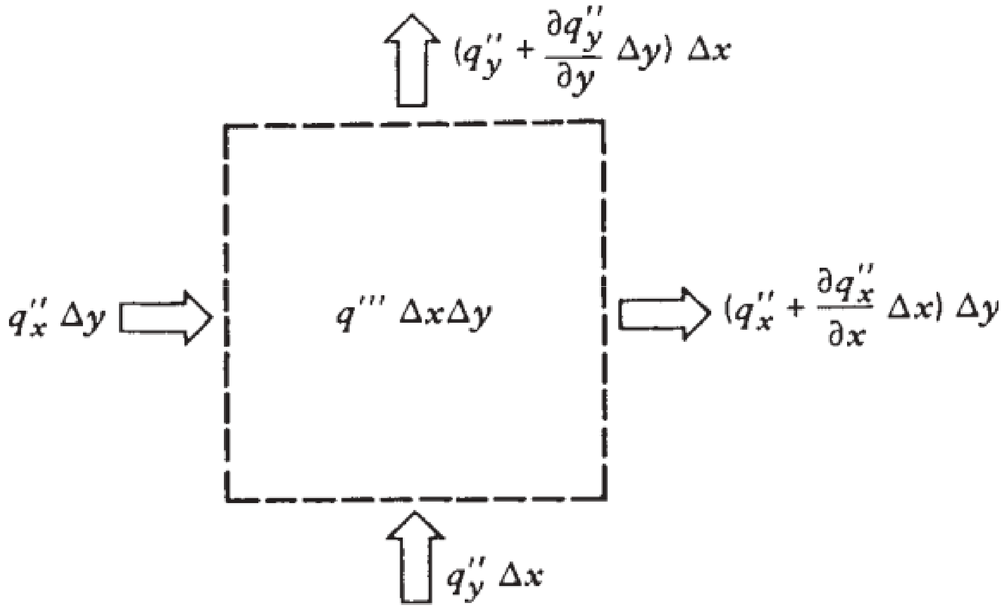
$$\frac{D}{Dt}=\frac{\partial}{\partial t}+v_r\frac{\partial}{\partial r}+\frac{v_\phi}{r}\frac{\partial}{\partial\phi}+\frac{v_\theta}{r\sin\phi}\frac{\partial}{\partial\theta}$$

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)+\frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}$$

FIRST LAW OF THERMODYNAMICS

$$\begin{pmatrix}\text{rate of energy}\\ \text{accumulation in the}\\ \text{control volume}\end{pmatrix}_1 = \begin{pmatrix}\text{net transfer of}\\ \text{energy by fluid flow}\end{pmatrix}_2 + \begin{pmatrix}\text{net heat transfer}\\ \text{by conduction}\end{pmatrix}_3 + \begin{pmatrix}\text{rate of internal}\\ \text{heat generation (e.g.,}\\ \text{electrical power}\\ \text{dissipation)}\end{pmatrix}_4 - \begin{pmatrix}\text{net work transfer}\\ \text{from the control}\\ \text{volume to its}\\ \text{environment}\end{pmatrix}_5$$





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$$\begin{pmatrix}\text{rate of energy}\\ \text{accumulation in the}\\ \text{control volume}\end{pmatrix}_1 = \begin{pmatrix}\text{net transfer of}\\ \text{energy by fluid flow}\end{pmatrix}_2 + \begin{pmatrix}\text{net heat transfer}\\ \text{by conduction}\end{pmatrix}_3 + \begin{pmatrix}\text{rate of internal}\\ \text{heat generation (e.g.,}\\ \text{electrical power}\\ \text{dissipation)}\end{pmatrix}_4 - \begin{pmatrix}\text{net work transfer}\\ \text{from the control}\\ \text{volume to its}\\ \text{environment}\end{pmatrix}_5$$

$$\{\cdot\}_1 = \Delta x\,\Delta y\,\frac{\partial}{\partial t}(\rho e) \qquad \{\cdot\}_2 = -(\Delta x\,\Delta y)\left[\frac{\partial}{\partial x}(\rho u e) + \frac{\partial}{\partial y}(\rho v e)\right]$$

$$\{\cdot\}_3 = -(\Delta x\,\Delta y)\left(\frac{\partial q_x''}{\partial x} + \frac{\partial q_y''}{\partial y}\right) \qquad \{\cdot\}_4 = (\Delta x\,\Delta y)q'''$$

$$\{\cdot\}_5 = (\Delta x\,\Delta y)\left(\sigma_x\frac{\partial u}{\partial x} - \tau_{xy}\frac{\partial u}{\partial y} + \sigma_y\frac{\partial v}{\partial y} - \tau_{yx}\frac{\partial v}{\partial x}\right) + (\Delta x\,\Delta y)\left(u\frac{\partial \sigma_x}{\partial x} - u\frac{\partial \tau_{xy}}{\partial y} + v\frac{\partial \sigma_y}{\partial y} - v\frac{\partial \tau_{yx}}{\partial x}\right)$$

$$\rho\frac{De}{Dt} + e\left(\frac{D\rho}{Dt} + \rho\,\nabla\cdot\mathbf{v}\right) = -\nabla\cdot\mathbf{q}'' + q''' - P\;\nabla\cdot\mathbf{v} + \mu\Phi$$

$\mathbf{q}''$ is the heat flux vector (q_x'', q_y'') and Φ is the *viscous dissipation function*

incompressible and two-dimensional ➡ $\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2$

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Thermodynamics definition $h = e + (1/\rho)P$ $\Rightarrow$ $\frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho}\frac{DP}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt}$

the *Fourier law of heat conduction* $\Rightarrow$ $\mathbf{q}'' = -k\,\nabla T$

$$\rho \times \left[\frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho}\frac{DP}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt} \right] \Rightarrow \rho\frac{De}{Dt} = \underbrace{\rho\frac{Dh}{Dt} - \frac{DP}{Dt} + \frac{P}{\rho}\frac{D\rho}{Dt}}$$

$$\rho\frac{De}{Dt} + e\left(\frac{D\rho}{Dt} + \rho\,\nabla\cdot\mathbf{v}\right) = -\nabla\cdot\mathbf{q}'' + q''' - P\;\nabla\cdot\mathbf{v} + \mu\Phi$$

$$\Rightarrow \rho\frac{Dh}{Dt} = \nabla\cdot(k\,\nabla T) + q''' + \frac{DP}{Dt} + \mu\,\Phi - \frac{P}{\rho}\underbrace{\left(\frac{D\rho}{Dt} + \rho\,\nabla\cdot\mathbf{v}\right)}_{=0}$$

$$\rho\frac{Dh}{Dt} = \nabla\cdot(k\,\nabla T) + q''' + \frac{DP}{Dt} + \mu\,\Phi$$

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Table 1.1 Summary of thermodynamic relations[a] and models

	Internal Energy $du = T\,ds - P\,dv$	Enthalpy $dh = T\,ds + v\,dP$	Entropy $ds = \frac{1}{T}du + \frac{P}{T}dv$
Pure substance	$du = c_v\,dT + \left[T\left(\frac{\partial P}{\partial T}\right)_v - P\right]dv$	$dh = c_p\,dT + \left[-T\left(\frac{\partial v}{\partial T}\right)_p + v\right]dP$	$ds = \frac{c_p}{T}dT - \left(\frac{\partial v}{\partial T}\right)_P dP = \frac{c_v}{T}dT + \left(\frac{\partial P}{\partial T}\right)_v dv$
Ideal gas	$du = c_v\,dT$	$dh = c_P\,dT$	$ds = c_p\frac{dT}{T} - R\frac{dP}{P} = c_v\frac{dT}{T} + R\frac{dv}{v} = c_v\frac{dP}{P} + c_P\frac{dv}{v}$
Incompressible liquid	$du = c\,dT$	$dh = c\,dT + v\,dP$	$ds = c\frac{dT}{T}$

$$dh = T\,ds + \frac{1}{\rho}\,dP \qquad ds = \left(\frac{\partial s}{\partial T}\right)_P dT + \left(\frac{\partial s}{\partial P}\right)_T dP$$

T is the absolute temperature

Maxwell's relations $$\left(\frac{\partial s}{\partial P}\right)_T = -\left[\frac{\partial\,(1/\rho)}{\partial T}\right]_P = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_P = -\frac{\beta}{\rho}$$

β is the coefficient of thermal expansion $$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P$$

$$\left(\frac{\partial s}{\partial T}\right)_P = \frac{c_P}{T}$$

$$dh = T\,ds + \frac{1}{\rho}\,dP$$

$$ds = \frac{c_p}{T}dT - \left(\frac{\partial v}{\partial T}\right)_P dP$$

$$dh = c_P\,dT + \frac{1}{\rho}(1 - \beta T)\,dP$$

$$\rho\frac{Dh}{Dt} = \rho c_P \frac{DT}{Dt} + (1 - \beta T)\frac{DP}{Dt}$$

$$\rho\frac{Dh}{Dt} = \nabla \cdot (k\,\nabla T) + q''' + \frac{DP}{Dt} + \mu\,\Phi$$

$$dh = c_p\,dT$$

$$\rho c_P \frac{DT}{Dt} = \nabla \cdot (k\,\nabla T) + q''' + \beta T\frac{DP}{Dt} + \mu\,\Phi$$

Ideal gas ($\beta = 1/T$): $$\rho c_P \frac{DT}{Dt} = \nabla \cdot (k \nabla T) + q''' + \frac{DP}{Dt} + \mu \Phi$$

Incompressible liquid ($\beta = 0$) $$\rho c \frac{DT}{Dt} = \nabla \cdot (k \nabla T) + q''' + \mu \Phi$$

negligible viscous dissipation $\mu\Phi$, and negligible compressibility effect βT

zero internal heat generation q''' → $$\rho c_P \frac{DT}{Dt} = k \nabla^2 T$$

Cartesian (x, y, z):

$$\rho c_P \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

Cylindrical (r, θ, z):

$$\rho c_P \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right]$$

Spherical (r, ϕ, θ):

$$\rho c_P \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\phi}{r} \frac{\partial T}{\partial \phi} + \frac{v_\theta}{r \sin \phi} \frac{\partial T}{\partial \theta} \right)$$

$$= k \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial T}{\partial \phi} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 T}{\partial \theta^2} \right]$$

the internal heating due to viscous dissipation $$\rho c_P \frac{DT}{Dt} = k \nabla^2 T + \mu \Phi$$

Cartesian (x, y, z):

$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2\right] - \frac{2}{3}\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2}$$

$\nabla \cdot \mathbf{v} = 0$

Cylindrical (r, θ, z)

$$\Phi = 2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)^2 + \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)^2 - \frac{1}{3}\underbrace{(\nabla \cdot \mathbf{v})^2}\right]$$

$\nabla \cdot \mathbf{v} = 0$

Spherical (r, ϕ, θ):

$$\Phi = 2\left\{\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r}\right)^2 + \left(\frac{1}{r\sin\phi}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{v_\phi \cot\phi}{r}\right)^2\right]\right.$$

$$+ \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \phi}\right]^2 + \frac{1}{2}\left[\frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\left(\frac{v_\theta}{r\sin\phi}\right) + \frac{1}{r\sin\phi}\frac{\partial v_\theta}{\partial \theta}\right]^2$$

$$\left.+\frac{1}{2}\left[\frac{1}{r\sin\phi}\frac{\partial v_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]^2\right\} - \frac{2}{3}\underbrace{(\nabla\cdot\mathbf{v})^2}_{\nabla\cdot\mathbf{v}=0}$$

SECOND LAW OF THERMODYNAMICS

The second law of thermodynamics states that all real-life processes are irreversible

$$\frac{\partial S_{cv}}{\partial t} \geq \sum \frac{q_i}{T_i} + \sum_{\substack{\text{inlet}\\\text{ports}}} \dot{m}s - \sum_{\substack{\text{outlet}\\\text{ports}}} \dot{m}s$$

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Defined as positive *into* the control volume

$$S_{gen} = \frac{\partial S_{cv}}{\partial t} - \sum \frac{q_i}{T_i} - \sum_{\substack{\text{inlet}\\ \text{ports}}} \dot{m}s + \sum_{\substack{\text{outlet}\\ \text{ports}}} \dot{m}s \geq 0$$

absolute temperature

$$W_{lost} = T_0 S_{gen}$$

T_0 is the absolute temperature of the ambient temperature reservoir (T_0 = constant)

the rate of entropy generation per unit time and per unit volume S'''_{gen} is

$$S'''_{gen} = \underbrace{\frac{k}{T^2}(\nabla T)^2}_{\geq 0} + \underbrace{\frac{\mu}{T}\Phi}_{\geq 0} \geq 0$$

In a two-dimensional

$$S'''_{gen} = \frac{k}{T^2}\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right] + \frac{\mu}{T}\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right\} \geq 0$$

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RULES OF SCALE ANALYSIS

example of scale analysis conduction heat transfer

a plate plunged at $t = 0$ into a highly conducting fluid

estimating the time needed by the thermal front to penetrate the plate

we focus on a half-plate of thickness $D/2$

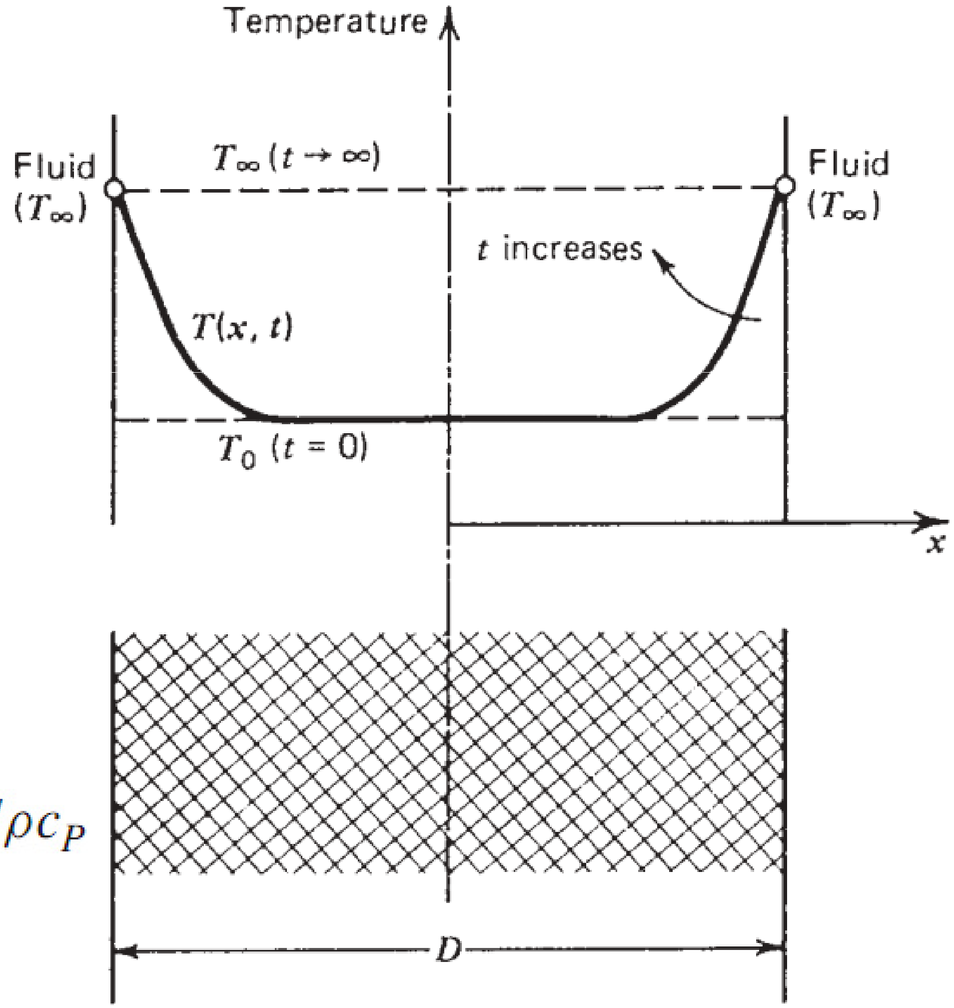
$$\rho c_P \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

$$\rho c_P \frac{\partial T}{\partial t} \sim \rho c_P \frac{\Delta T}{t}$$

$$k \frac{\partial^2 T}{\partial x^2} = k \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) \sim \frac{k}{D/2}\frac{\Delta T}{D/2} = \frac{k\,\Delta T}{(D/2)^2}$$

$$t \sim \frac{(D/2)^2}{\alpha}$$

α is the thermal diffusivity of the medium, $k/\rho c_P$



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rules of scale analysis

- *Rule 1*. Always define the spatial extent of the region in which you perform the scale analysis.

- *Rule 2*. One equation constitutes an equivalence between the scales of two dominant terms appearing in the equation.

- *Rule 3*. If in the sum of two terms, $c = a + b$

$$O(a) > O(b) \Rightarrow O(c) = O(a) \quad \left[O(c) \sim O(a) \right]$$

The same conclusion holds $\Rightarrow$ $c = a - b$ or $c = -a + b$.

- *Rule 4*. If in the sum of two terms, $c = a + b$, the two terms are of the same order of magnitude, $O(a) = O(b) \Rightarrow O(c) \sim O(a) \sim O(b)$

- *Rule 5*. In any product $p = ab \Rightarrow O(p) = O(a)O(b)$

$$r = \frac{a}{b} \Rightarrow O(r) = \frac{O(a)}{O(b)}$$

In rules 1–5, we used the following symbols:

$\sim$	is of the same order of magnitude as
$O(a)$	the order of magnitude of a
$>$	greater than, in an order-of-magnitude sense

HEATLINES FOR VISUALIZING CONVECTION

streamfunction $\psi(x, y)$ → $u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$

$H(x, y)$ *heatfunction* H

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

or

$$\frac{\partial}{\partial x}\left(\rho c_P u T - k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho c_P v T - k\frac{\partial T}{\partial y}\right) = 0$$

The heatfunction is defined as follows:

Net energy flow in the x direction: $\frac{\partial H}{\partial y} = \rho c_P u(T - T_{\text{ref}}) - k\frac{\partial T}{\partial x}$

Net energy flow in the y direction: $-\frac{\partial H}{\partial x} = \rho c_P v(T - T_{\text{ref}}) - k\frac{\partial T}{\partial y}$

Shohel Mahmud, Roydon Andrew Frase, ***Visualizing energy flows through energy streamlines and pathlines***, *International Journal of Heat and Mass Transfer 50 (2007) 3990–4002*

The top wall in Fig. 1 is hot and is moving from left to right while the remaining three walls are cold and stationary.

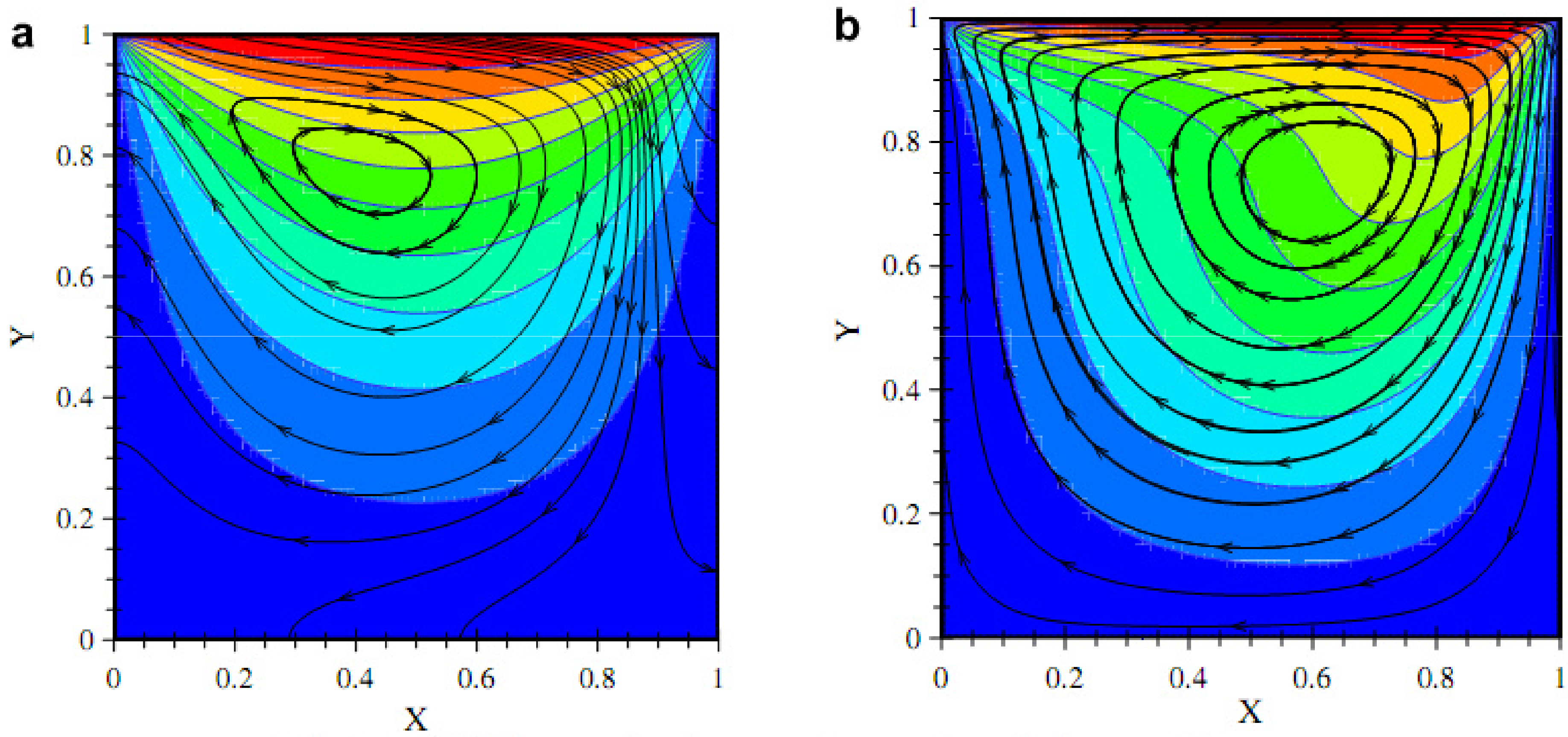


Fig. 1. Lid driven cavity flow at (a) $Re = 1$ and (b) $Re = 100$.

The arrow-marks in each energy streamline indicate its direction

The next example is the classical buoyancy induced natural convection in a square cavity problem the two vertical walls are isothermal but the temperature of the left wall is higher than the right wall, while the top and bottom walls are thermally insulated

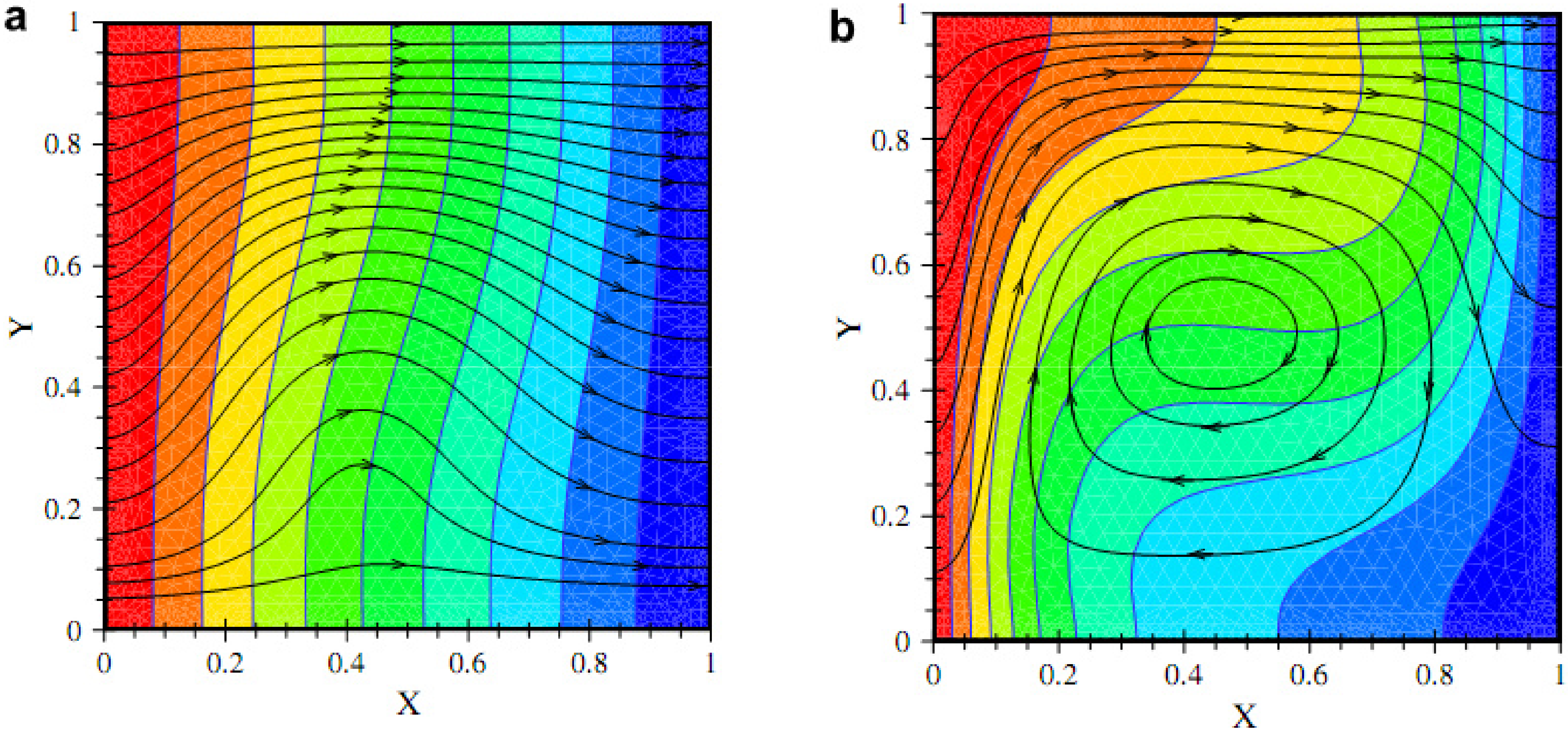
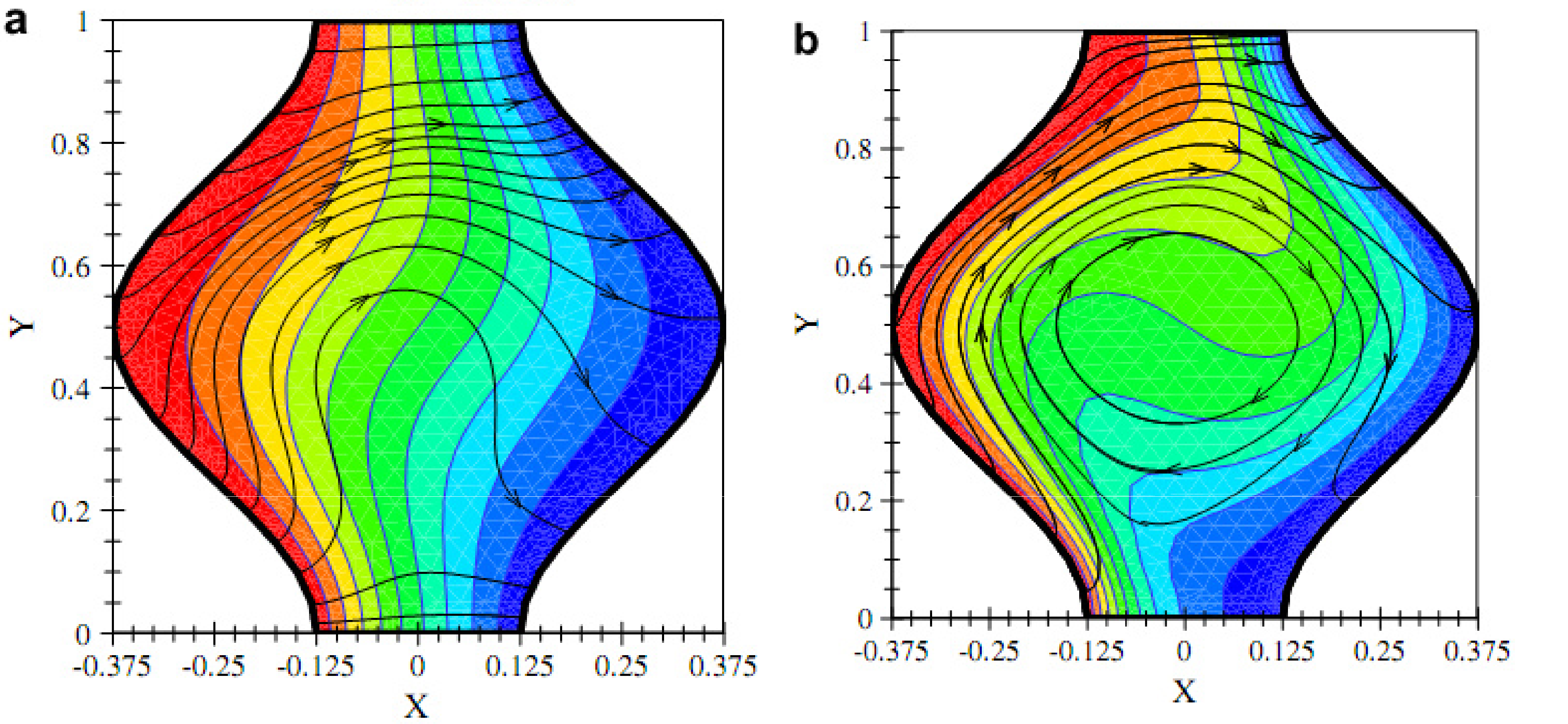


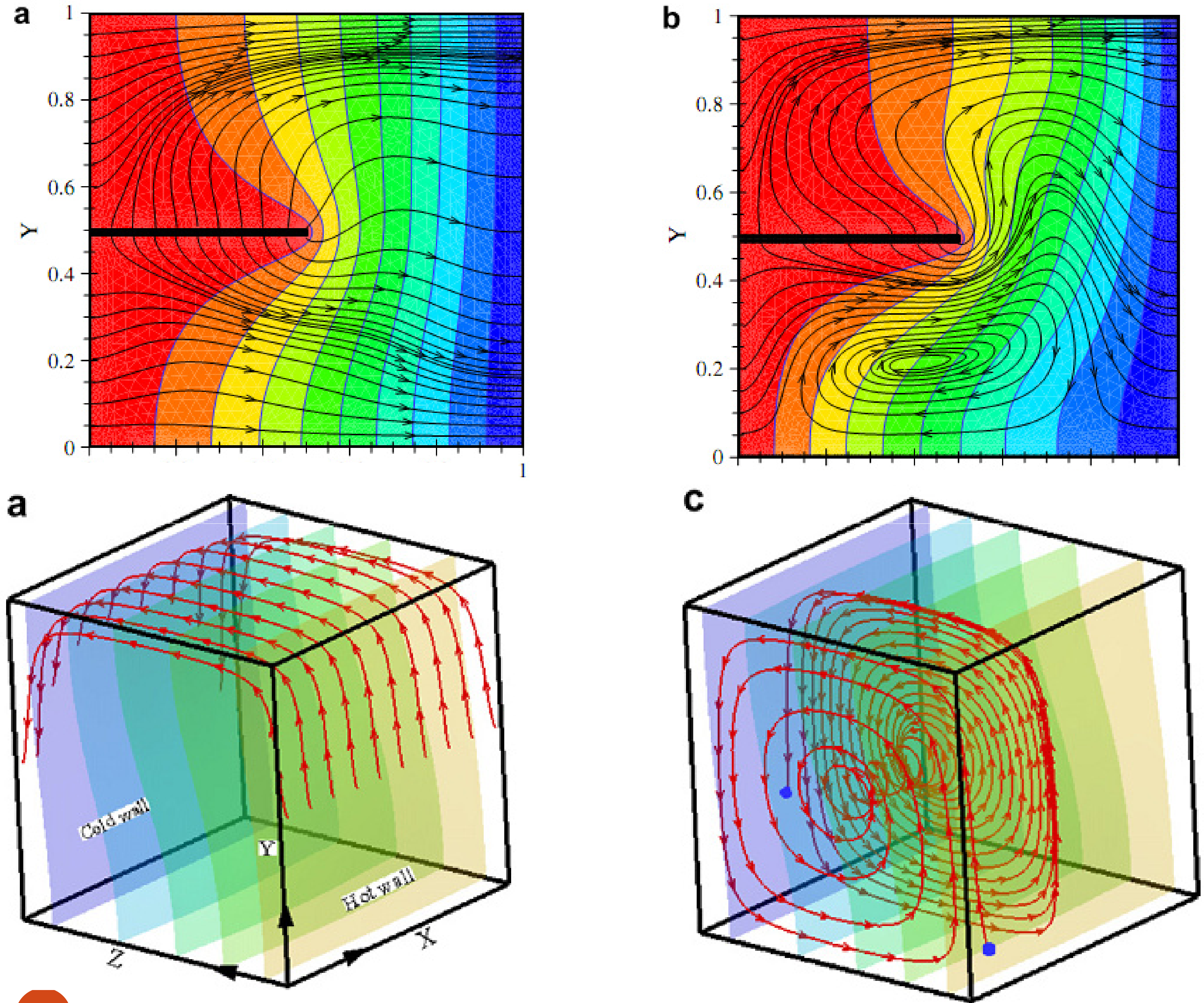
Fig. 2. Buoyancy induced flow in a square cavity at (a) $Ra = 5 \times 10^2$ and (b) $Ra = 1 \times 10^4$.

By: M. Farhadi, Faculty of Mechanical Engineering, Babol University of Technology

free convection

Fig. 4. Free convection inside a wavy cavity at (a) $Ra = 5 \times 10^2$ and (b) $Ra = 1 \times 10^4$.




a
b
Y
1
0.8
0.6
0.4
0.2
0
1
a
c
Cold wall
Hot wall
X
Y
Z