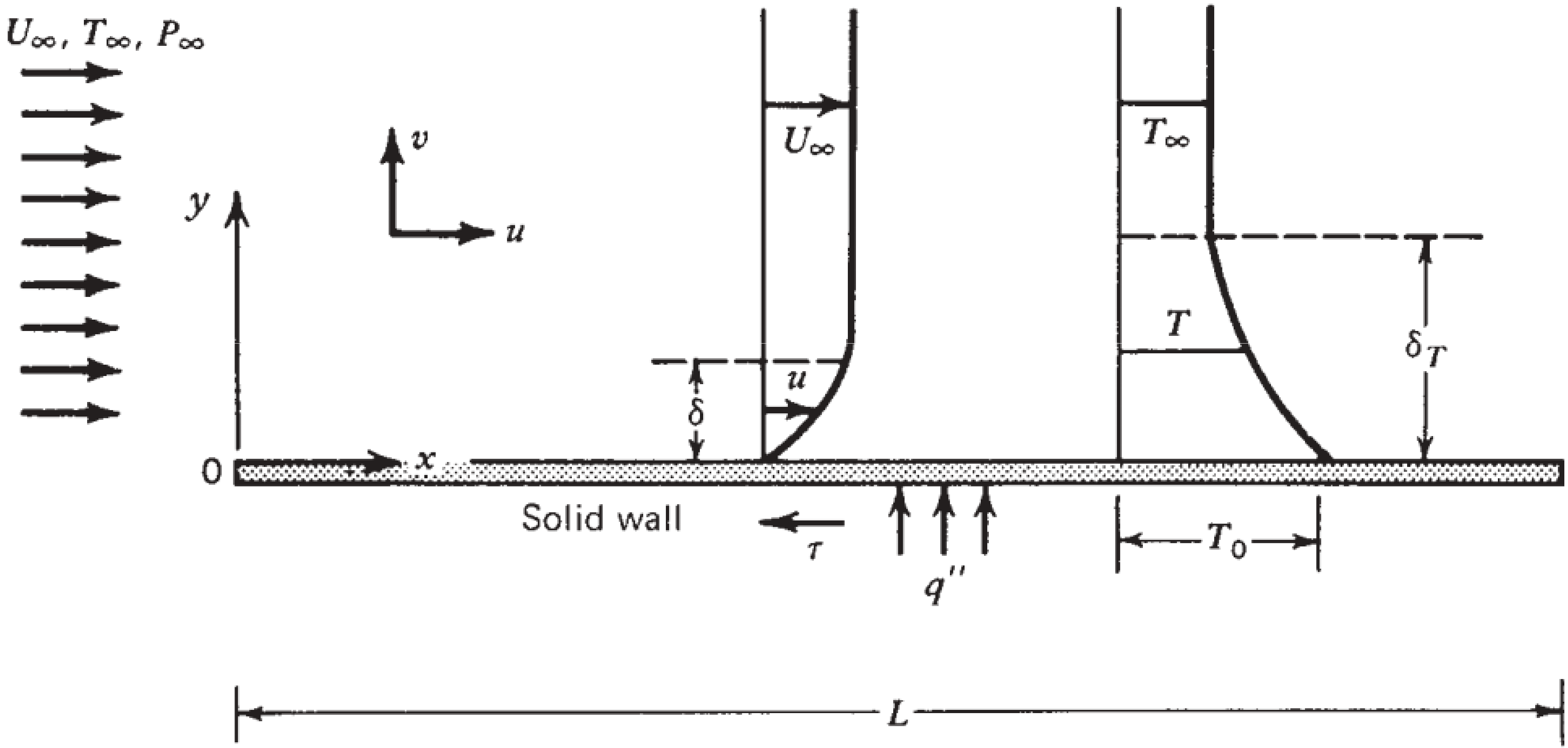
LAMINAR BOUNDARY LAYER FLOW



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we are interested in calculating the

the total force $F = \int_0^L \tau W\, dx$ → $\tau = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$

the total heat transfer rate $q = \int_0^L q'' W\, dx$ → $q'' = h(T_0 - T_\infty)$

at $y = 0^+$ → *no-slip hypothesis*

heat from the wall to the fluid is first by pure conduction $q'' = -k \left(\frac{\partial T}{\partial y}\right)_{y=0}$

$$\left.\begin{aligned} q'' &= -k \left(\frac{\partial T}{\partial y}\right)_{y=0} \\ q'' &= h(T_0 - T_\infty) \end{aligned}\right\} h = \frac{-k(\partial T/\partial y)_{y=0}}{T_0 - T_\infty}$$

→ برای پاسخ به سوالات فوق لازم است تا گرادیان
سرعت و دما در کنار دیوار محاسبه گردد

Modeling the flow as incompressible and of constant property (Chapter 1), the complete mathematical statement of this problem consists of the following. Solve four equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial v} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

four unknowns (u, v, P, T),

boundary conditions

(i)	No slip	$u = 0$	at the solid wall
(ii)	Impermeability	$v = 0$	
(iii)	Wall temperature	$T = T_0$	
(iv)	Uniform flow	$u = U_\infty$	infinitely far from the solid, in both the y and x directions
(v)	Uniform flow	$v = 0$	
(vi)	Uniform temperature	$T = T_\infty$	

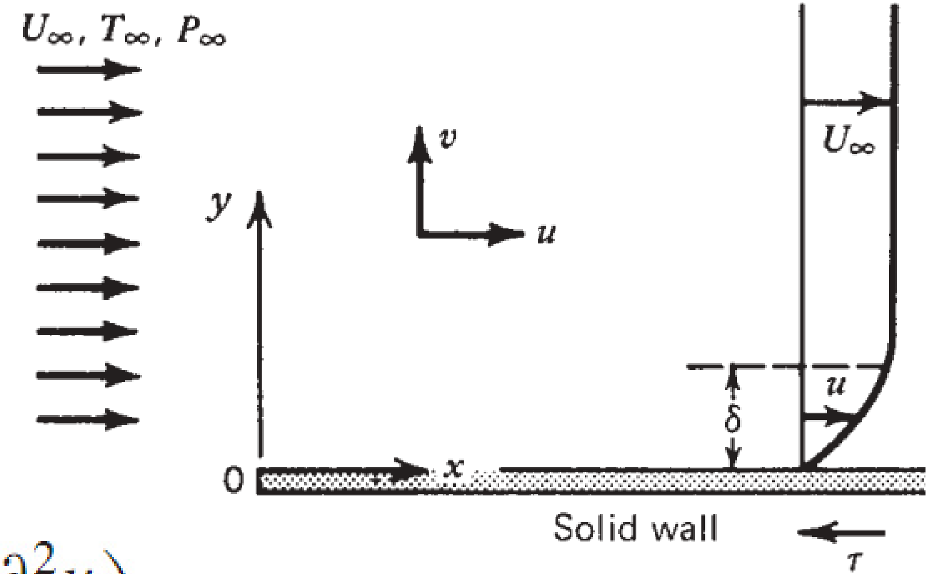
The free stream is characterized by

$$u = U_\infty, \quad v = 0, \quad P = P_\infty, \quad T = T_\infty$$

the scale analysis

$$x \sim L, \quad y \sim \delta \qquad u \sim U_\infty$$

In the $\delta \times L$ region



$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Inertia Pressure Friction

$$U_\infty \frac{U_\infty}{L}, \quad v\frac{U_\infty}{\delta} \qquad \frac{P}{\rho L} \qquad \nu\frac{U_\infty}{L^2}, \quad \nu\frac{U_\infty}{\delta^2}$$

?

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{U_\infty}{L} \sim \frac{v}{\delta} \Rightarrow v \sim \delta\frac{U_\infty}{L}$$

$$\Rightarrow v\frac{U_\infty}{\delta} = U_\infty\frac{U_\infty}{L}$$

$\delta \ll L$

$$\nu\frac{U_\infty}{L^2} \ll \nu\frac{U_\infty}{\delta^2}$$

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$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

The free stream $P = P_\infty$,

the y momentum equation reduces to

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\frac{\partial^2 v}{\partial y^2}$$

$$v \sim \delta\frac{U_\infty}{L}, \quad \delta \ll L \quad \Rightarrow \quad v \ll u$$

in the x momentum equation
the pressure ~ friction ➡ $\dfrac{\partial P}{\partial x} \sim \dfrac{\mu U_\infty}{\delta^2}$

the y momentum equation $\dfrac{\partial P}{\partial y} \sim \dfrac{\mu v}{\delta^2}$

$$\frac{(\partial P/\partial y)(dy/dx)}{\partial P/\partial x} \sim \frac{v\delta}{U_\infty L} \sim \left(\frac{\delta}{L}\right)^2 \ll 1$$

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy \;\Rightarrow\; \frac{dP}{dx} = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}\frac{dy}{dx}$$

$$\frac{dP}{dx} = \frac{\partial P}{\partial x} \qquad \frac{\partial P}{\partial x} = \frac{dP_\infty}{dx}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

the x momentum equation

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$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \Rightarrow \frac{\partial T}{\partial x}\sim\frac{\Delta T}{L},\quad \frac{\partial T}{\partial y}\sim\frac{\Delta T}{\delta_t},\quad \frac{\partial^2 T}{\partial x^2}\sim\frac{\Delta T}{L^2},\quad \frac{\partial^2 T}{\partial y^2}\sim\frac{\Delta T}{\delta^2}$$

$$\Delta T = T_0 - T_\infty$$

$$u = U_\infty,\ \upsilon\sim\frac{U_\infty\delta_t}{L} \Rightarrow \frac{\partial T}{\partial x}\sim\frac{U_\infty\Delta T}{L},\quad \frac{\partial T}{\partial y}\sim\frac{\upsilon\Delta T}{\delta_t}=\frac{U_\infty\Delta T}{L}$$

$$\frac{\partial^2 T}{\partial x^2}\sim\frac{\Delta T}{L^2},\quad \frac{\partial^2 T}{\partial y^2}\sim\frac{\Delta T}{\delta_t^2} \Rightarrow \frac{\partial^2 T}{\partial y^2}\gg\frac{\partial^2 T}{\partial x^2}$$

$$\Rightarrow \boxed{u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}}$$

The boundary layer equation for energy

$$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0 \\ u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}=-\dfrac{1}{\rho}\dfrac{dP_\infty}{dx}+\nu\dfrac{\partial^2 u}{\partial y^2} \\ u\dfrac{\partial T}{\partial x}+v\dfrac{\partial T}{\partial y}=\alpha\dfrac{\partial^2 T}{\partial y^2} \end{cases}$$

معادلات لایه مرزی که باید برای جریان آرام بر روی یک صفحه حل گردند

SCALE ANALYSIS

$\delta \neq \delta_T.$

$$\tau = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} \rightarrow \tau \sim \mu \frac{U_\infty}{\delta}$$

free stream with uniform pressure P_∞

With $dP_\infty / dx = 0$ $\rightarrow$ $u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$ $\rightarrow$ $\frac{U_\infty^2}{L}, \frac{v U_\infty}{\delta} \sim \nu \frac{U_\infty}{\delta^2}$

inertia $\sim$ friction

$$\delta \sim \left(\frac{\nu L}{U_\infty} \right)^{1/2}$$

$$Re_L = U_\infty L / \nu$$

$$\boxed{\frac{\delta}{L} \sim \mathrm{Re}_L^{-1/2}}$$

The time of transversal viscous diffusion $t_\delta \sim \delta^2 / \nu$

The time of longitudinal convection $t_L \sim L / U_\infty$

at the trailing end of the wall these two times are in fact the same time scale, $t_\delta \sim t_L$

$$\delta \ll L \rightarrow \mathrm{Re}_L^{1/2} \gg 1$$

نتایج برای حالت فوق صحیح می باشد

$$\tau \sim \mu \frac{U_\infty}{\delta}$$

$$\frac{\delta}{L} \sim \mathrm{Re}_L^{-1/2}$$

$$\tau \sim \mu \frac{U_\infty}{L} \mathrm{Re}_L^{1/2} \sim \rho U_\infty^2 \mathrm{Re}_L^{-1/2}$$

skin friction coefficient $C_f = \tau / \left(\frac{1}{2}\rho U_\infty^2\right)$

$$C_f \sim \mathrm{Re}_L^{-1/2}$$

$$h = \frac{-k(\partial T/\partial y)_{y=0}}{T_0 - T_\infty}$$

$$\frac{\partial T}{\partial y} \sim \frac{\Delta T}{\delta_T}$$

$$h \sim \frac{k(\Delta T/\delta_T)}{\Delta T} \sim \frac{k}{\delta_T} \quad \rightarrow \quad h \sim k/\delta_T$$

$$\Delta T = T_0 - T_\infty$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

the region $\delta_T \times L$

convection ~ conduction

$$u\frac{\Delta T}{L}, v\frac{\Delta T}{\delta_T} \sim \alpha \frac{\Delta T}{\delta_T^2}$$

در این بررسی با توجه به وجود سرعت ، مقایسه ضخامت لایه مرزی
هیدرولیکی و حرارتی ضروری است.

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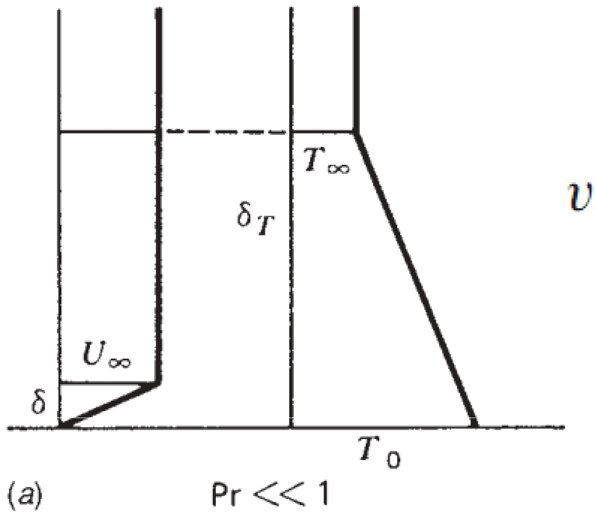
1. *Thick thermal boundary layer,* $\delta_T \gg \delta$.

$$u\frac{\Delta T}{L}, v\frac{\Delta T}{\delta_T} \sim \alpha\frac{\Delta T}{\delta_T^2}$$

u scale outside the velocity boundary layer (and inside the δ_T layer) is U_∞



$$v \sim U_\infty \delta/L$$

$$(u\ \Delta T)/L \rightarrow (U_\infty\ \Delta T)/L$$

$$v\frac{\Delta T}{\delta_T} \sim U_\infty\frac{\Delta T}{L}\frac{\delta}{\delta_T}$$

the convection ~ conduction balance

$$\delta/\delta_T \ll 1$$

$$(U_\infty\ \Delta T)/L \sim (\alpha\ \Delta T)/\delta_T{}^2$$

$$\rightarrow \frac{\delta_T}{L} \sim \mathrm{Pe}_L^{-1/2} \sim \mathrm{Pr}^{-1/2}\,\mathrm{Re}_L^{-1/2}$$

$$\frac{\delta_T}{\delta} \sim \mathrm{Pr}^{-1/2} \gg 1$$

$\mathrm{Pe}_L = U_\infty L/\alpha$ Péclet number

Prandtl number $\mathrm{Pr} = \nu/\alpha$

The first assumption, $\delta_T \gg \delta$, is therefore valid in the limit $\mathrm{Pr}^{1/2} \ll 1$

liquid metals

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$$h \sim k/\delta_T$$

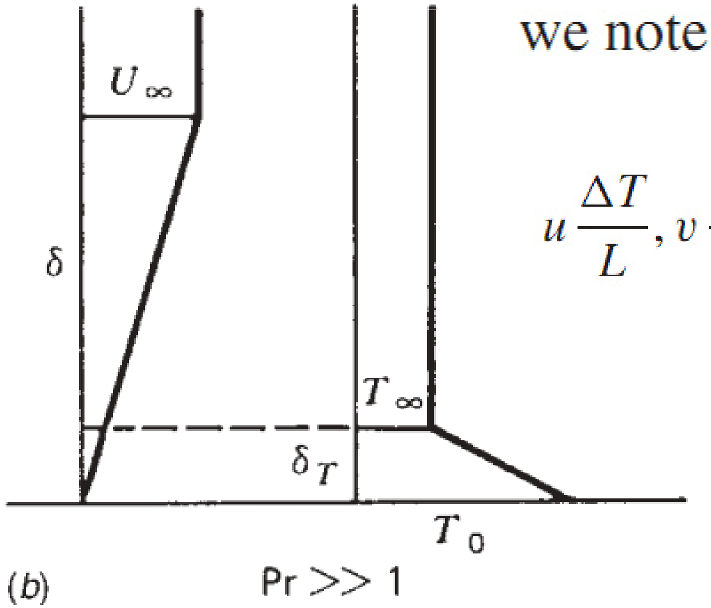
$$\frac{\delta_T}{L} \sim \mathrm{Pe}_L^{-1/2} \sim \mathrm{Pr}^{-1/2}\,\mathrm{Re}_L^{-1/2}$$

$$h \sim \frac{k}{L}\mathrm{Pr}^{1/2}\,\mathrm{Re}_L^{1/2} \qquad (\mathrm{Pr} \ll 1)$$

$$\mathrm{Nu} = hL/k \;\rightarrow\; \boxed{\mathrm{Nu} \sim \mathrm{Pr}^{1/2}\,\mathrm{Re}_L^{1/2}}$$

2. *Thin thermal boundary layer,* $\delta_T \ll \delta$

it is clear that the scale of u in the δ_T layer is not U_∞ but $u \sim U_\infty \dfrac{\delta_T}{\delta}$



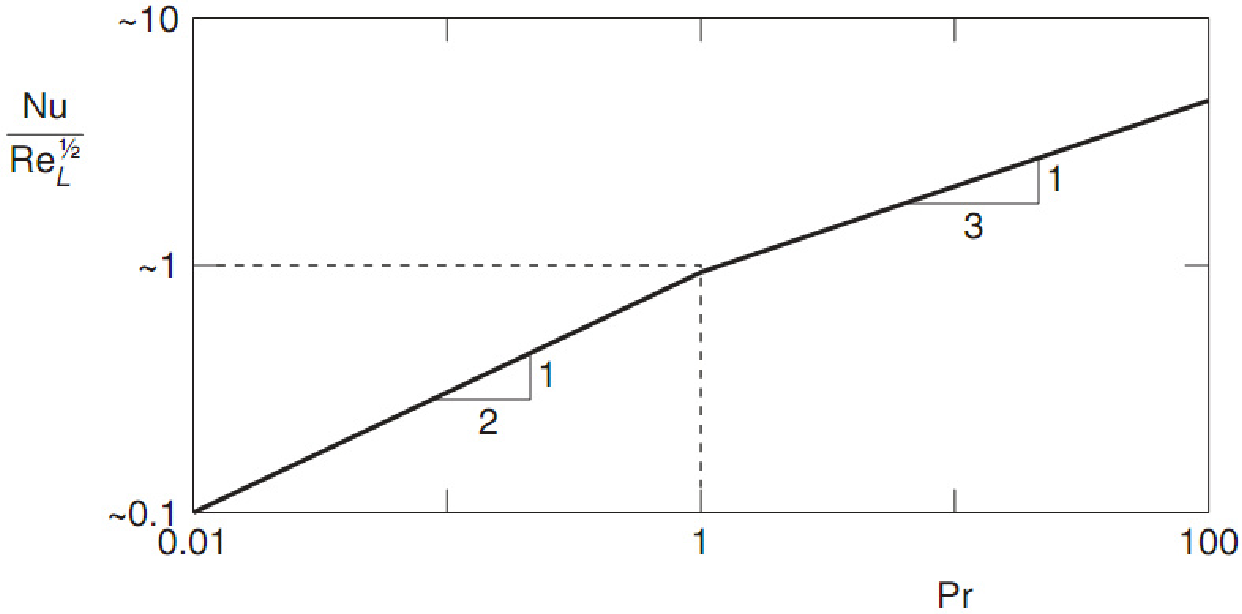
we note that $u/L \sim v/\delta_T$ because of mass conservation

$$u\frac{\Delta T}{L},\; v\frac{\Delta T}{\delta_T} \sim \alpha\frac{\Delta T}{\delta_T^2} \;\rightarrow\; \left.\begin{array}{l} u/L \sim \alpha/\delta_T^2 \\ u \sim U_\infty \dfrac{\delta_T}{\delta} \\ \dfrac{\delta}{L} \sim \mathrm{Re}_L^{-1/2} \end{array}\right\} \frac{\delta_T}{L} \sim \mathrm{Pr}^{-1/3}\,\mathrm{Re}_L^{-1/2}$$

$$\frac{\delta_T}{\delta} \sim \mathrm{Pr}^{-1/3} \ll 1$$

$$h \sim \frac{k}{L}\mathrm{Pr}^{1/3}\,\mathrm{Re}_L^{1/2} \qquad (\mathrm{Pr} \gg 1)$$

$$\boxed{\mathrm{Nu} \sim \mathrm{Pr}^{1/3}\,\mathrm{Re}_L^{1/2} \qquad (\mathrm{Pr} \gg 1)}$$



The meaning of Reynolds number → magnitude of the inertia/friction ratio

In the boundary layer → **there is always a balance between inertia and friction**

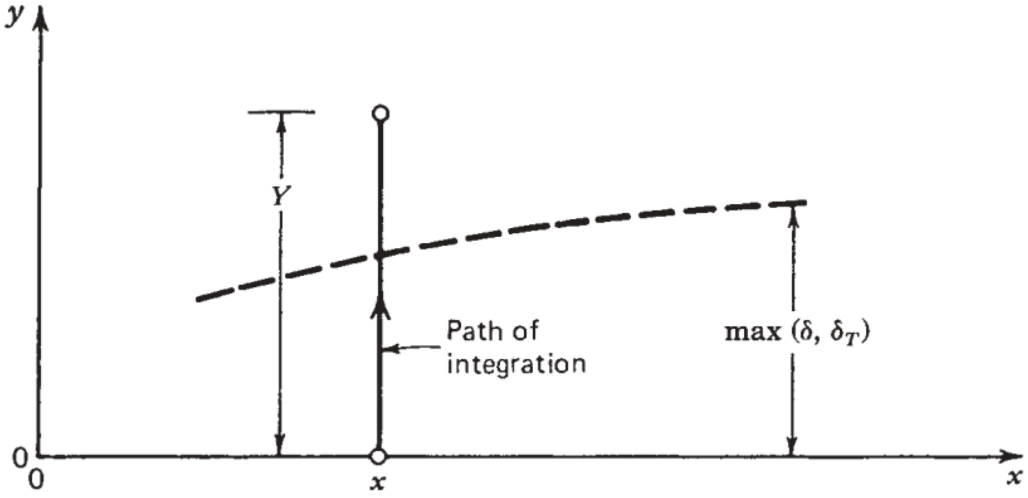
The only physical interpretation of the Reynolds number in boundary layer flow is geometric:

$$\mathrm{Re}_L^{1/2} = \frac{\text{wall length}}{\text{boundary layer thickness}}$$

it is not Re_L, but the square root of Re_L, that means something:

$\mathrm{Re}_L^{1/2}$ is *a geometric parameter of the flow region*—the *slenderness ratio*

INTEGRAL SOLUTIONS

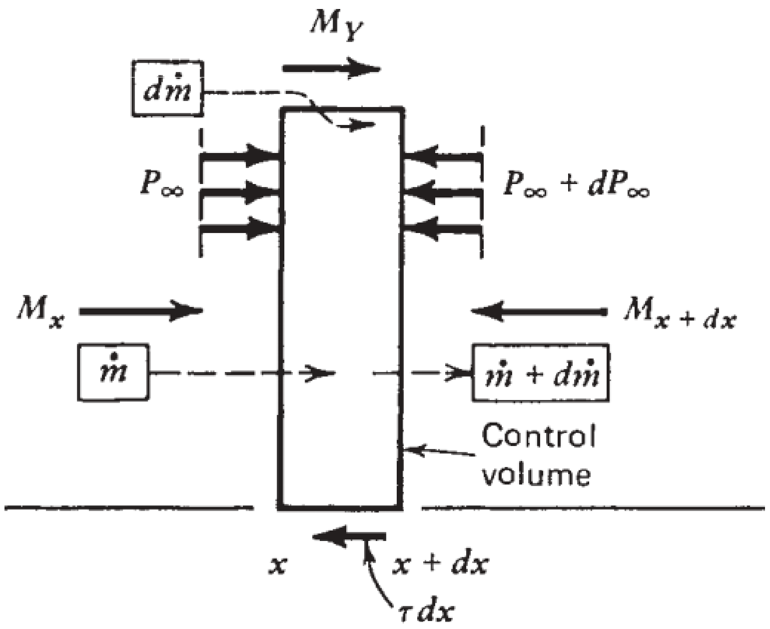


$$\tau_{0-L} = \frac{1}{L}\int_0^L \tau\, dx, \qquad h_{0-L} = \frac{1}{L}\int_0^L h\, dx$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$



by integrating each equation term by term from $y = 0$ to $y = Y$, where $Y > \max(\delta,\delta_T)$ is situated in the free stream

Before integrating

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) = \alpha\frac{\partial^2 T}{\partial y^2}$$

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$$\frac{d}{dx}\int_0^Y u^2\,dy + u_Y v_Y - u_0 v_0 = -\frac{1}{\rho} Y \frac{dP_\infty}{dx} + \nu\left(\frac{\partial u}{\partial y}\right)_Y - \nu\left(\frac{\partial u}{\partial y}\right)_0$$

$$\frac{d}{dx}\int_0^Y uT\,dy + v_Y T_Y - v_0 T_0 = \alpha\left(\frac{\partial T}{\partial y}\right)_Y - \alpha\left(\frac{\partial T}{\partial y}\right)_0$$

Because the free stream is uniform, we note that $(\partial/\partial y)_Y = 0$ $u_Y = U_\infty$, and $T_Y = T_\infty$

$$v_0 = 0$$

integral on the continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ ➡ $\frac{d}{dx}\int_0^Y u\,dy + v_Y - v_0 = 0$

$$v_Y = -\frac{d}{dx}\int_0^Y u\,dy$$

با جایگذاری در رابطه مومنتم در لایه مرزی:

$$\frac{d}{dx}\int_0^Y u(U_\infty - u)\,dy = \frac{1}{\rho} Y \frac{dP_\infty}{dx} + \frac{dU_\infty}{dx}\int_0^Y u\,dy + \nu\left(\frac{\partial u}{\partial y}\right)_0$$

$$\frac{d}{dx}\int_0^Y u(T_\infty - T)\,dy = \frac{dT_\infty}{dx}\int_0^Y u\,dy + \alpha\left(\frac{\partial T}{\partial y}\right)_0$$

the integral boundary layer equations for momentum and energy

$$M_x = \int_0^Y \rho u^2 \, dy \qquad M_Y = U_\infty \, d\dot{m}$$

$P_\infty Y$ Force due to pressure

where $\dot{m} = \int_0^Y \rho u \, dy$ is the mass flow rate through the slice of height Y

$$M_{x+dx} = M_x + (dM_x/dx)\, dx$$

$\tau \, dx$ Tangential force due to friction

$Y[P_\infty + (dP_\infty/dx)\, dx]$ Force due to pressure

$$\frac{d}{dx}\int_0^Y u(U_\infty - u)\, dy = \frac{1}{\rho} Y \frac{dP_\infty}{dx} + \frac{dU_\infty}{dx}\int_0^Y u \, dy + \nu \left(\frac{\partial u}{\partial y}\right)_0$$

با همین روش و با استفاده از قانون اول ترمودینامیک می توان به رابطه انتگرالی انرژی در لایه مرزی رسید

در روش انتگرالی تابع سرعت و دما باید به عنوان اطلاعات ورودی در نظر گرفته شود و سایر نتایج بر اساس این توابع، محاسبه گردد.

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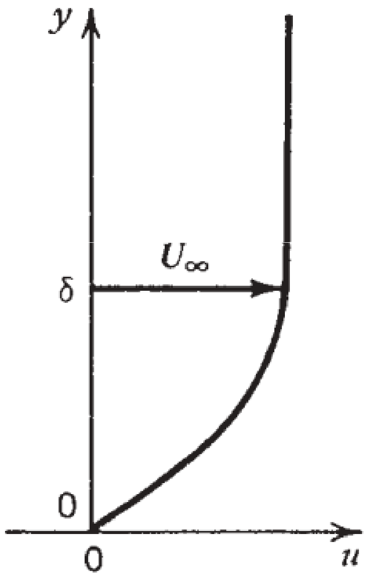
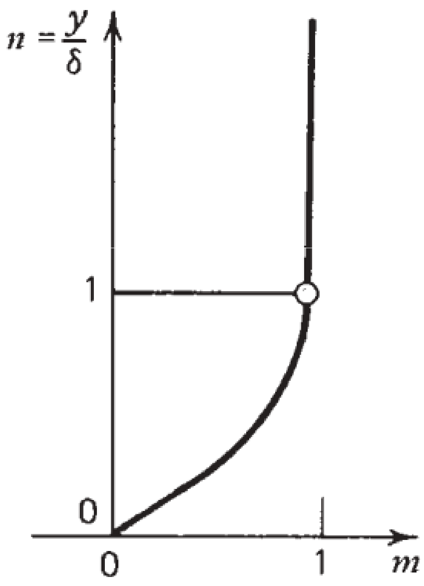
the *shape* of the longitudinal velocity profile is described by

$$u = \begin{cases} U_\infty m(n), & 0 \le n \le 1 \\ U_\infty, & 1 \le n \end{cases} \qquad n = y/\delta$$

m is an unspecified shape function that varies from 0 to 1





assumption $\begin{cases} dP_\infty/dx = 0 \\ dU_\infty/dx = 0 \end{cases}$

$$\delta \frac{d\delta}{dx}\left[\int_0^1 m(1-m)\,dn\right] = \frac{\nu}{U_\infty}\left(\frac{dm}{dn}\right)_{n=0}$$

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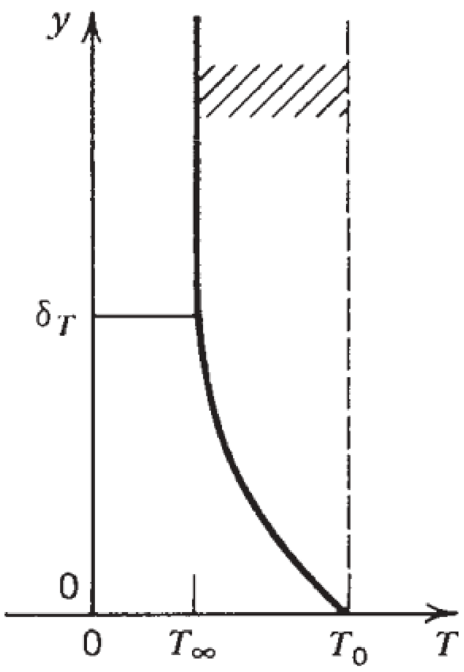
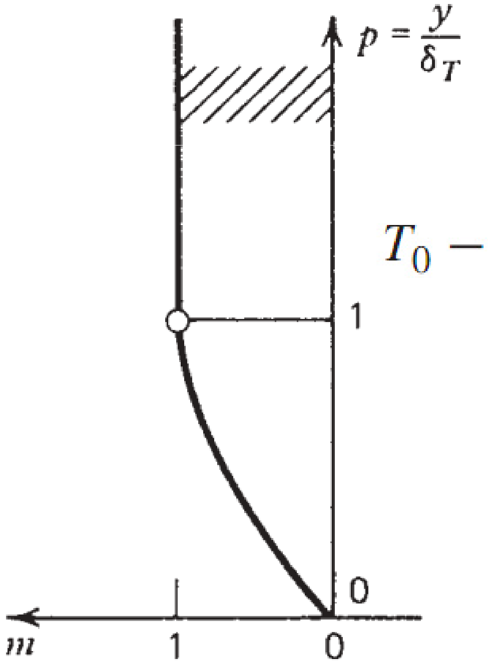
$$\frac{\delta}{x} = a_1 \, \mathrm{Re}_x^{-1/2} \qquad a_1 = \left[\frac{2(dm/dn)_{n=0}}{\int_0^1 m(1-m)\, dn} \right]^{1/2}$$

$$C_{f,x} = \frac{\tau}{\frac{1}{2}\rho U_\infty^2} = a_2 \, \mathrm{Re}_x^{-1/2} \qquad a_2 = \left[2\left(\frac{dm}{dn}\right)_{n=0} \int_0^1 m(1-m)\, dn \right]^{1/2}$$





$$dT_\infty/dx = 0 \qquad p = y/\delta_T$$

$$T_0 - T = (T_0 - T_\infty)m(p), \qquad 0 \le p \le 1$$

$$T = T_\infty, \qquad 1 \le p$$

$$\frac{\delta_T}{\delta} = \Delta$$

$\delta_T < \delta$ (high-Pr fluids)

$$\frac{d}{dx}\int_0^Y u(T_\infty - T)\,dy = \frac{dT_\infty}{dx}\int_0^Y u\,dy + \alpha\left(\frac{\partial T}{\partial y}\right)_0$$

$$\mathrm{Pr} = \frac{2(dm/dp)_{p=0}}{(a_1\Delta)^2}\left[\int_0^1 m\,(p\Delta)\,[1 - m(p)]\,dp\right]^{-1}$$

$$dT_\infty/dx = 0 \qquad p = y/\delta_T$$

$$T_0 - T = (T_0 - T_\infty)m(p), \; 0 \le p \le 1$$

$$T = T_\infty, \qquad 1 \le p$$

$\frac{\delta_T}{\delta} = \Delta$ is a function of Prandtl number only

Table 2.1 Impact of the assumed profile shape on the integral solution to the laminar boundary layer friction and heat transfer problem

Profile Shape $m(n)$ or $m(p)$ (Fig. 2.4)	$\frac{\delta}{x}\mathrm{Re}_x^{1/2}$	$C_{f,x}\,\mathrm{Re}_x^{1/2}$	$\mathrm{Nu}\,\mathrm{Re}_x^{-1/2}\,\mathrm{Pr}^{-1/3}$ Uniform Temperature (Pr > 1)	$\mathrm{Nu}\,\mathrm{Re}_x^{-1/2}\,\mathrm{Pr}^{-1/3}$ Uniform Heat Flux (Pr > 1)
$m = n$	3.46	0.577	0.289	0.364
$m = (n/2)(3 - n^2)$	4.64	0.646	0.331	0.417
$m = \sin(\pi n/2)$	4.8	0.654	0.337	0.424
Similarity solution	4.92[a]	0.664	0.332	0.453

Assuming the simplest temperature profile, $m = p$,

$$T_0 - T = (T_0 - T_\infty) m(p), \qquad 0 \le p \le 1$$

$$T = T_\infty, \qquad 1 \le p$$

$$\mathrm{Pr} = \frac{2(dm/dp)_{p=0}}{(a_1 \Delta)^2} \left[\int_0^1 m(p\Delta)\,[1 - m(p)]\,dp \right]^{-1}$$

$\Delta = \mathrm{Pr}^{-1/3}$

for $\mathrm{Pr} \gg 1$ fluids

the cubic profile $m = (p/2)(3 - p^2)$ ➡ $\Delta = \dfrac{\delta_T}{\delta} = 0.976\mathrm{Pr}^{-1/3}$

$$h = 0.331 \frac{k}{x} \mathrm{Pr}^{1/3} \mathrm{Re}_x^{1/2} \quad \Longrightarrow \quad \mathrm{Nu} = \frac{hx}{k} = 0.331 \mathrm{Pr}^{1/3} \mathrm{Re}_x^{1/2}$$

In the case of liquid metals ($\Delta \gg 1$)

$$\mathrm{Pr} = \frac{2(dm/dp)_{p=0}}{(a_1 \Delta)^2} \left[\int_0^{1/\Delta} m(p\Delta)\,[1 - m(p)]\,dp + \int_{1/\Delta}^1 [1 - m(p)]\,dp \right]^{-1}$$

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$\delta_T \gg \delta$, next to the wall $(0 < y < \delta)$, $U_\infty m$,

whereas for $\delta < y < \delta_T$, the velocity is uniform, $u = U_\infty$

$$\mathrm{Pr} = \frac{2(dm/dp)_{p=0}}{(a_1\Delta)^2}\left[\int_0^{1/\Delta} m\,(p\Delta)\,[1 - m(p)]\,dp + \int_{1/\Delta}^{1}[1 - m(p)]\,dp\right]^{-1}$$

the simplest profile $m = p$, ➡ $\Delta = \dfrac{\delta_T}{\delta} = (3\mathrm{Pr})^{-1/2} \qquad (\mathrm{Pr} \ll 1)$

$$\frac{\delta_T}{x} = 2\mathrm{Pr}^{-1/2}\,\mathrm{Re}_x^{-1/2} \qquad (\mathrm{Pr} \ll 1)$$

$$h = \frac{k}{\delta_T}$$

$$h = \frac{1}{2}\frac{k}{x}\mathrm{Pr}^{1/2}\,\mathrm{Re}_x^{1/2} \qquad (\mathrm{Pr} \ll 1)$$

$$\mathrm{Nu} = \frac{hx}{k} = \frac{1}{2}\mathrm{Pr}^{1/2}\,\mathrm{Re}_x^{1/2} \qquad (\mathrm{Pr} \ll 1)$$

SIMILARITY SOLUTIONS

$$\frac{u}{U_\infty} = \text{function}(\eta)$$ the *similarity variable* η is proportional to y

that η must be proportional to $y/\delta(x)$

with $\delta \sim x\,\text{Re}_x^{-1/2}$

$$\frac{u}{U_\infty} = f'(\eta), \qquad \eta = \frac{y}{x}\text{Re}_x^{1/2}$$

Function $f' = df/d\eta$ is presently unknown

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$P_\infty =$ constant boundary layer →

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

three boundary conditions

$$u = v = 0 \qquad \text{at } y = 0$$

$$u \to U_\infty \qquad \text{as } y \to \infty$$

the streamfunction $\psi(x,y)$ $$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

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$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}$$

$$\frac{\partial \psi}{\partial y} = 0, \quad \psi = 0 \text{ at } y = 0$$

$$\frac{\partial \psi}{\partial y} \to U_\infty \quad \text{as } y \to \infty$$

$$u = \frac{\partial \psi}{\partial y},$$

$$\frac{u}{U_\infty} = f'(\eta), \qquad \eta = \frac{y}{x}\mathrm{Re}_x^{1/2}$$

$$\psi = (U_\infty \nu x)^{1/2} f(\eta) \;\Rightarrow\; v = \frac{1}{2}\left(\frac{\nu U_\infty}{x}\right)^{1/2}(\eta f' - f)$$

جایگذاری در معادله مومنتم

$$2f''' + ff'' = 0$$

boundary conditions

$$f' = f = 0 \quad \text{at } \eta = 0$$

$$f' \to 1 \quad \text{as } \eta \to \infty$$

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$$u = 0.99U_\infty \text{ at } \eta = 4.92$$

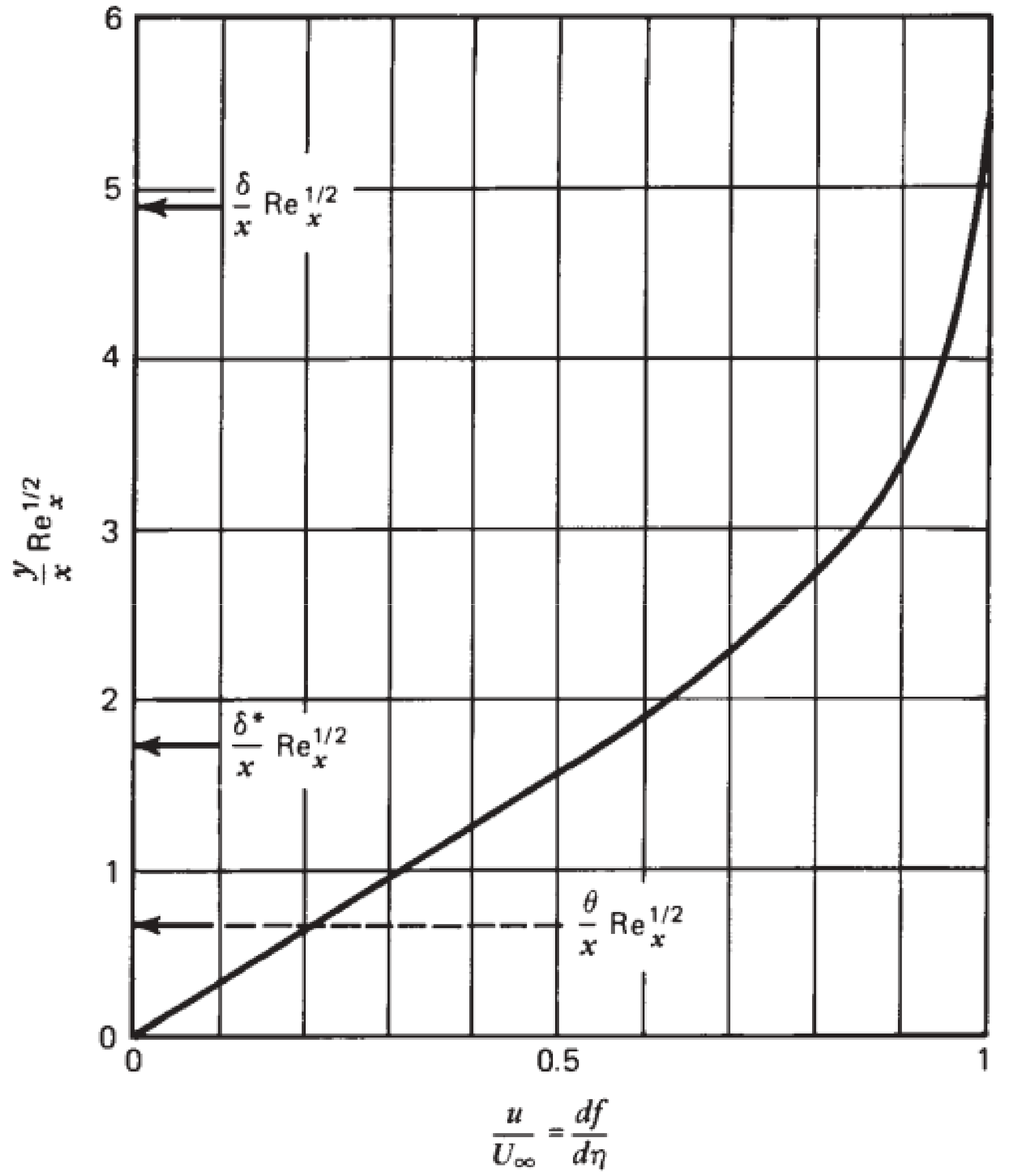
$$\frac{\delta}{x} = 4.92\mathrm{Re}_x^{-1/2}$$

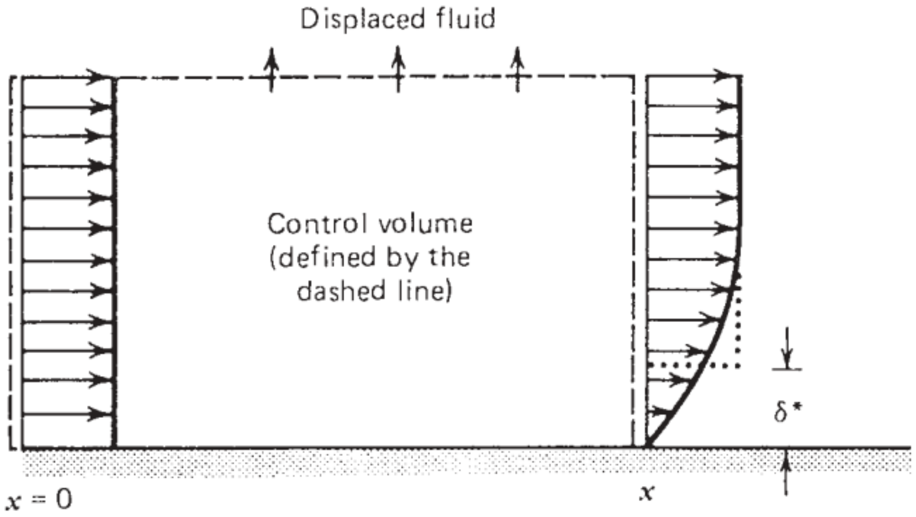
Displacement thickness:

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy$$

Momentum thickness:

$$\theta = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$





Displacement thickness:

$$\delta^* U_\infty = \int_0^\infty U_\infty \, dy - \int_0^\infty u \, dy$$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy$$

Momentum thickness:

$$\theta U_\infty^2 = \underbrace{\int_0^\infty U_\infty^2 \, dy}_{x \text{ momentum at } x=0} - \underbrace{\int_0^\infty U^2 \, dy}_{x \text{ momentum at any } x} - \underbrace{U_\infty \int_0^\infty (U_\infty - u) \, dy}_{x \text{ momentum of the fluid displaced out of the boundary layer region}}$$

$$\Rightarrow \quad \theta = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

similarity solution $\quad \dfrac{\delta^*}{x} = 1.73 \mathrm{Re}_x^{-1/2}, \qquad \dfrac{\theta}{x} = 0.664 \mathrm{Re}_x^{-1/2}$

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local skin friction coefficient $C_{f,x} = \dfrac{\mu(\partial u/\partial y)_0}{\frac{1}{2}\rho U_\infty^2} = 2(f'')_{\eta=0}\,\mathrm{Re}_x^{-1/2}$

Numerically, it is found that $(f'')_{y=0} = 0.332$ ➡ $C_{f,x} = 0.664\mathrm{Re}_x^{-1/2}$

average skin friction coefficient $C_{f,0-x} = \dfrac{\tau_{0-x}}{\frac{1}{2}\rho U_\infty^2} = 1.328\mathrm{Re}_x^{-1/2} = 2\,C_{f,x}$

valid when $\mathrm{Re}_x \lesssim 5 \times 10^5$

Heat Transfer Solution

$$\theta(\eta) = \frac{T - T_0}{T_\infty - T_0} \qquad u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

$$\frac{u}{U_\infty} = f'(\eta), \qquad \eta = \frac{y}{x}\mathrm{Re}_x^{1/2}$$

$$\theta'' + \frac{\mathrm{Pr}}{2} f\theta' = 0$$

$$\theta = 0 \quad \text{at } \eta = 0$$

$$\theta \to 1 \quad \text{as } \eta \to \infty$$

Note that if $\mathrm{Pr} = 1$ and $\theta = f'$.

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integrate eq. $\theta'' + \frac{\Pr}{2} f\theta' = 0$ → $\theta'(\eta) = \theta'(0) \exp\left[-\frac{\Pr}{2}\int_0^\eta f(\beta)\,d\beta\right]$

Integrating again from 0 to η and using the wall condition $\theta = 0$ at $\eta = 0$

?

$$\theta(\eta) = \theta'(0)\int_0^\eta \exp\left[-\frac{\Pr}{2}\int_0^\gamma f(\beta)\,d\beta\right]d\gamma$$

where β and γ are two dummy variables

as $\eta \to \infty$ → $\theta \to 1$ → $\theta'(0) = \left\{\int_0^\infty \exp\left[-\frac{\Pr}{2}\int_0^\gamma f(\beta)\,d\beta\right]d\gamma\right\}^{-1}$

$\theta'(0)$ is a function of the Prandtl number

$$h = \frac{-k(\partial T/\partial y)_{y=0}}{T_0 - T_\infty}$$

$$\theta(\eta) = \frac{T - T_0}{T_\infty - T_0}$$

$$\eta = \frac{y}{x}\mathrm{Re}_x^{1/2}$$

$h = \frac{k}{x}\mathrm{Re}_x^{1/2}\,\theta'(0)$ → $\mathrm{Nu} = \frac{hx}{k} = \theta'(0)\,\mathrm{Re}_x^{1/2}$

Pohlhausen [21] calculated several $\theta'(0)$ values that for $\Pr > 0.5$

$\theta'(0) = 0.332\Pr^{1/3}$ → $\mathrm{Nu} = 0.332\Pr^{1/3}\mathrm{Re}_x^{1/2}$ $(\Pr > 0.5)$

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$$\Pr \to 0. \quad \left.\begin{array}{l} \dfrac{u}{U_\infty} = f'(\eta) \\ u = U_\infty \end{array}\right\} f' = 1$$

$$\theta'' + \frac{\Pr}{2} f\theta' = 0 \rightarrow \left(\frac{\theta''}{\theta'}\right) = -\frac{\Pr}{2} f \rightarrow \frac{d}{d\eta}\left(\frac{\theta''}{\theta'}\right) = -\frac{\Pr}{2} f'$$

$$\theta(\eta) = \operatorname{erf}\left(\frac{\eta}{2}\Pr^{1/2}\right)$$

$$\theta'(0) = \left(\frac{\Pr}{\pi}\right)^{1/2}$$

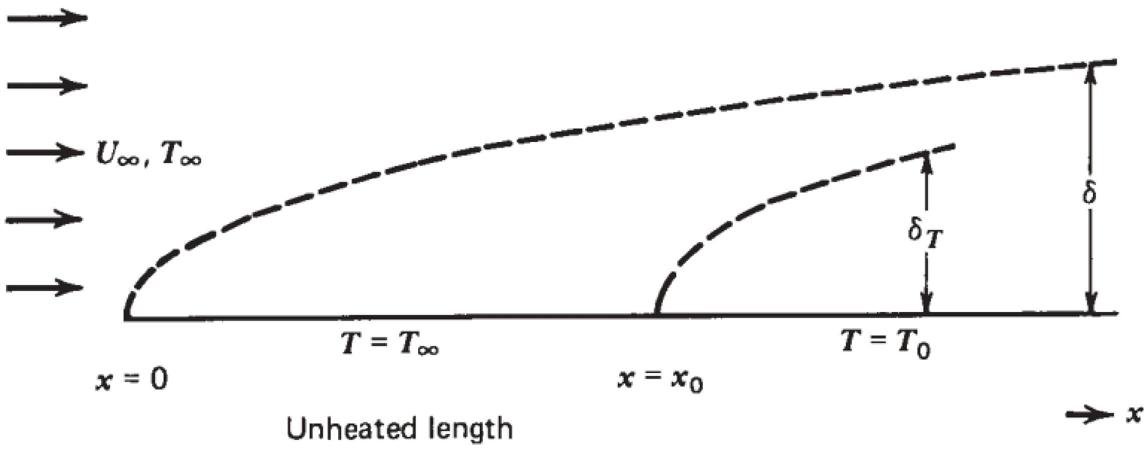
$$\mathrm{Nu} = \frac{hx}{k} = 0.564\Pr^{1/2}\mathrm{Re}_x^{1/2} \quad (\Pr \to 0)$$

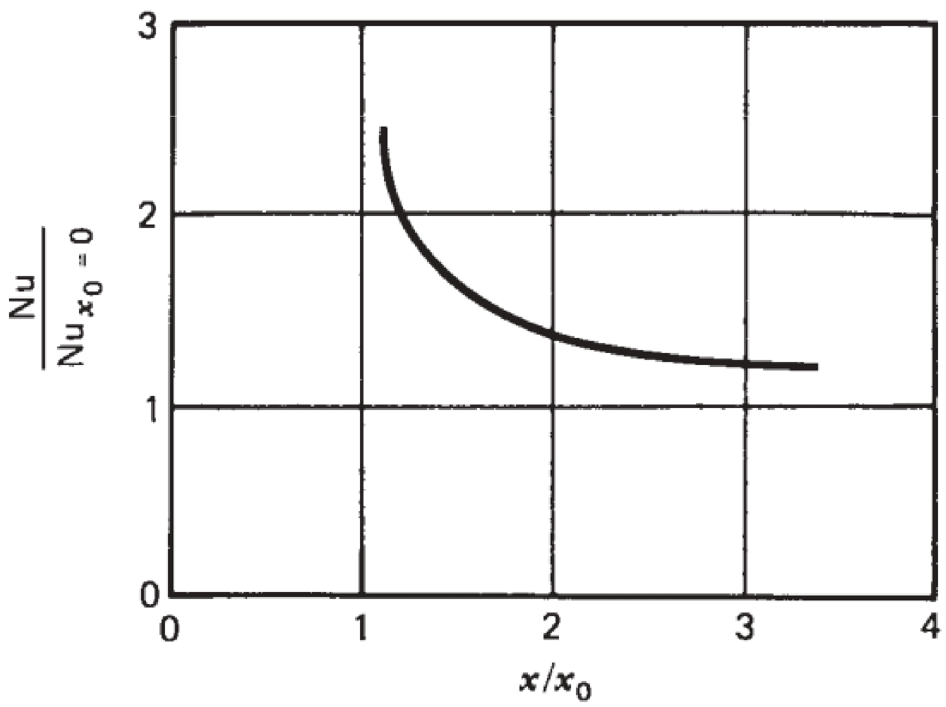
the average heat transfer coefficient $\mathrm{Nu}_{0-x} = \begin{cases} 0.664\Pr^{1/3}\mathrm{Re}_x^{1/2} & (\Pr > 0.5) \\ 1.128\Pr^{1/2}\mathrm{Re}_x^{1/2} & (\Pr < 0.5) \end{cases}$

Churchill and Ozoe [24]: $\mathrm{Nu}_{0-x} = \dfrac{0.928\Pr^{1/3}\mathrm{Re}_x^{1/2}}{[1 + (0.0207/\Pr)^{2/3}]^{1/4}}$

It is valid when the Péclet number $\mathrm{Pe}_x = U_\infty x/\alpha = \mathrm{Re}_x \Pr$ is greater than approximately 100

Unheated Starting Length





based on the integral method

Assuming the temperature profile shape

$$m = (p/2)(3 - p^2)$$

the velocity cubic profile shape

$$m = (n/2)(3 - n^2)$$

integral energy equation

$$\Delta^3 + 4\Delta^2 x \frac{d\Delta}{dx} = \frac{0.929}{\text{Pr}}$$

with the general solution

Constant

$$\Delta^3 = \frac{0.929}{\text{Pr}} + Cx^{-3/4}$$

$$\Delta = 0.976\text{Pr}^{-1/3}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}$$

$$\text{Nu} = \frac{hx}{k} = 0.332\text{Pr}^{1/3}\,\text{Re}_x^{1/2}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{-1/3}$$

Arbitrary Wall Temperature

دلخواه

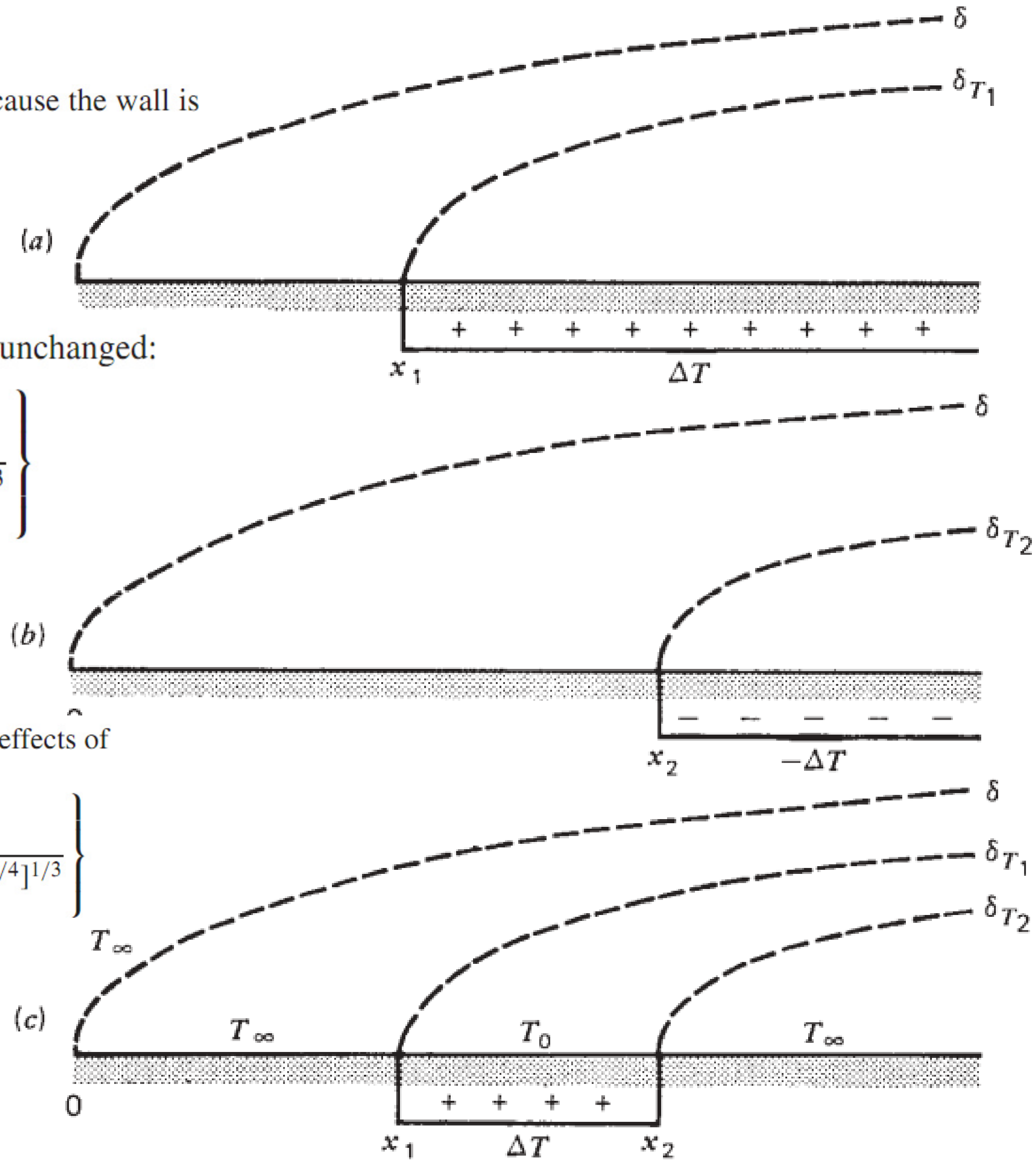
1. $0 < x < x_1$, the unheated started length, where $q'' = 0$ because the wall is in thermal equilibrium with the free stream

2. $x_1 < x < x_2$, the heated spot, where eq. (2.116) applies unchanged:

$$q'' = 0.332 \frac{k}{x} \Pr^{1/3} \mathrm{Re}_x^{1/2} \left\{ \frac{\Delta T}{\left[1 - (x_1/x)^{3/4}\right]^{1/3}} \right\}$$

3. $x > x_2$, the trailing section, where q'' is the superposition of two effects of type (2.117):

$$q'' = 0.332 \frac{k}{x} \Pr^{1/3} \mathrm{Re}_x^{1/2} \left\{ \frac{\Delta T}{\left[1 - (x_1/x)^{3/4}\right]^{1/3}} + \frac{-\Delta T}{[1 - (x_2/x)^{3/4}]^{1/3}} \right\}$$



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$$q'' = 0.332\,\frac{k}{x}\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}\,\sum_{i=1}^{N}\frac{\Delta T_i}{[1-(x_i/x)^{3/4}]^{1/3}}$$ N step changes ΔT_i in wall temperature

$$q'' = 0.332\,\frac{k}{x}\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}\,\int_0^x \frac{(dT_0/d\xi)\,d\xi}{[1-(\xi/x)^{3/4}]^{1/3}}$$ (the limit of infinitesimally small steps)

Uniform Heat Flux

$h = q''/[T_0(x) - T_\infty]$ when the heat flux q'' is known

the cubic profile $m = (p/2)(3 - p^2)$

$$\mathrm{Nu} = \frac{q''}{T_0(x) - T_\infty}\,\frac{x}{k} = 0.453\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2} \qquad (0.5 < \mathrm{Pr} < 10)$$

nonuniform wall heat flux $q''(x)$

$$T_0(x) - T_\infty = \frac{0.623}{k}\,\mathrm{Pr}^{-1/3}\,\mathrm{Re}_x^{-1/2}\,\int_{\xi=0}^{x}\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{-2/3} q''(\xi)\,d\xi$$

$$(\mathrm{Pr} > 0.5)$$

In real situations, fluid properties such as k, ν, μ, and α are not constant, they depend primarily on the local temperature

the constant-property formulas describe sufficiently accurately the actual convective flows encountered in applications, provided that

the fluid $(T_0 - T_\infty)$ is small

dimensionless groups (Re, Pe, Pr, C_f, Nu) can be evaluated at the average temperature of the fluid in the thermal boundary layer, $T = \frac{1}{2}(T_0 + T_\infty)$ *film temperature*

LONGITUDINAL PRESSURE GRADIENT: FLOW PAST A WEDGE AND STAGNATION FLOW

The potential flow solution for the velocity variation along the wedge-shaped wall is: $U_\infty(x) = Cx^m$

where C is a constant and m is related to the β angle of Fig. 2.10

$$m = \frac{\beta}{2\pi - \beta}$$

the Bernoulli equation along the streamline that coincides with the wall,

$$\frac{1}{\rho}\frac{dP_\infty}{dx} = -U_\infty \frac{dU_\infty}{dx} \Rightarrow \frac{1}{\rho}\frac{dP_\infty}{dx} = \frac{m}{x}U_\infty^2 \Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{m}{x}U_\infty^2 + \nu\frac{\partial^2 u}{\partial y^2}$$

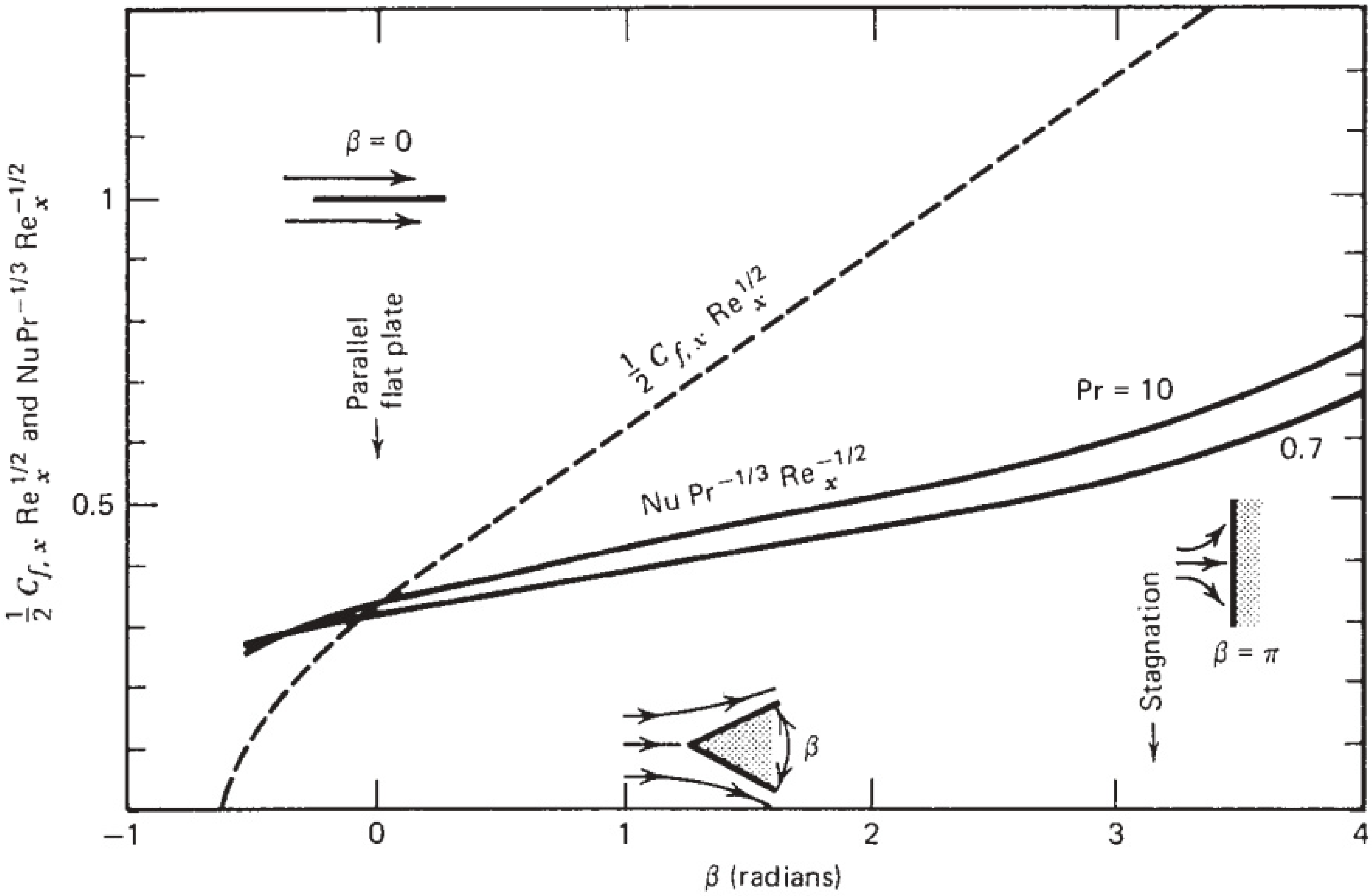

Figure 2.10 Local heat transfer and friction results for laminar boundary layer flow over an isothermal wedge-shaped body.

Falkner and Skan showed that → similarity solution $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{m}{x}U_\infty^2 + \nu\frac{\partial^2 u}{\partial y^2}$

$$2f''' + (m+1)ff'' + 2m[1-(f')^2] = 0$$

$\eta = y(U_\infty/\nu x)^{1/2}$ $U_\infty = Cx^m$ $u = U_\infty f'$ $f(0) = 0, f'(0) = 0$ $f'(\infty) = 1$

$C_{f,x} = 2f''(0)\,\mathrm{Re}_x^{-1/2}$

ایده حل بر اساس حل تشابهی بلازیوس با در نظر گرفتن m=0 بدست آمده است

Table 2.2 The local skin friction coefficient for laminar boundary layer flow over a wedge

β	m	$f''(0) = \frac{1}{2}C_{f,x}\mathrm{Re}_x^{1/2}$
$2\pi = 6.28$	∞	∞
$\pi = 3.14$	1	1.233 (two-dimensional stagnation)
$\pi/2 = 1.57$	$\frac{1}{3}$	0.757
$\pi/5 = 0.627$	$\frac{1}{9}$	0.512
0	0	0.332
−0.14	−0.0654	0.164
−0.199	−0.0904	0 (separation)

$\mathrm{Re}_x = U_\infty x/\nu = Cx^{m+1}/\nu$

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$$\theta(\eta) = (T - T_0)/(T_\infty - T_0) \Rightarrow \theta'' + \tfrac{1}{2}\,\mathrm{Pr}(m+1)f\theta' = 0 \qquad \theta(0) = 0 \text{ and } \theta(\infty) = 1$$

Table 2.3 Local Nusselt number $Nu/Re_x^{1/2}$ for laminar boundary layer flow over a wedge

		Pr				
β	m	0.7	0.8	1	5	10
−0.512	−0.0753	0.242	0.253	0.272	0.457	0.570
0	0	0.292	0.307	0.332	0.585	0.730
π/5	$\frac{1}{9}$	0.331	0.348	0.378	0.669	0.851
π/2	$\frac{1}{3}$	0.384	0.403	0.440	0.792	1.013
π	1	0.496	0.523	0.570	1.043	1.344
8π/5	4	0.813	0.858	0.938	1.736	2.236

The solid curves in Fig. 2.10 show that $Nu/Pr^{1/3}\,Re_x^{1/2}$ depends mainly on the wedge angle β

h varies as $x^{-1}\,Re_x^{1/2}$

$Re_x = U_\infty x/\nu = Cx^{m+1}/\nu$

h varies as $x^{(m-1)/2}$

$$h_{0-x} = \frac{h}{1 + (m-1)/2} = \frac{2}{1+m}h$$

$$Nu_x = \sqrt{\frac{m+1}{2}}\sqrt{Re_x}\,\theta'\Big|_{\eta^*=0} \qquad \alpha^* = \sqrt{\frac{m+1}{2}}\theta'\Big|_{\eta^*=0} \Rightarrow Nu_x = \alpha^*\sqrt{Re_x}$$

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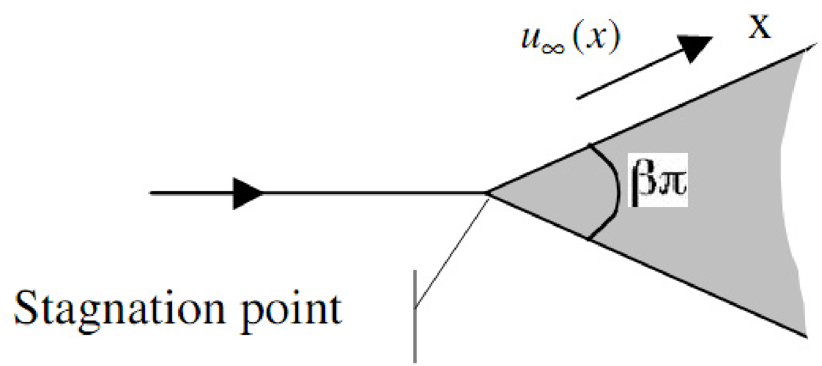
for $m = 0$ → flow over a flat plate.

A special case $\beta = \pi$ (or $m = 1$) In this case, U_∞ increases

For $m = 1$ $\alpha^*_{m=1} = 0.57 Pr^{0.4}$ → $Nu_x = 0.57 Pr^{0.4} Re_x^{0.5}$



$$h(x) = k\sqrt{\frac{V_\infty(x)}{\nu x}}\,\frac{d\theta(0)}{d\eta}.$$

$$\frac{d\theta(0)}{d\eta} = \left\{\int_0^\infty \exp\left[-\frac{(m+1)Pr}{2}\int_0^\eta F(\eta)d\eta\right] d\eta\right\}^{-1}.$$

$$u(x,\infty) = V_\infty(x) = Cx^m.$$

For $m = 1$

h is a constant.

More detail:
Latif M. Jiji, Heat Convection, 2006, pp. 143-149

FLOW THROUGH THE WALL: BLOWING AND SUCTION

where the boundary layer fluid crosses the wall surface with the normal velocity $v_0(x)$,

Positive v_0 values indicate *blowing*, Negative v_0 values represent *suction*,

the free stream $U_\infty = Cx^m$ The surface is isothermal (T_0).

$v = -\partial\psi/\partial\xi$ with $\psi = (U_\infty \nu x)^{1/2} f(\eta)$, $\eta = y(U_\infty/\nu x)^{1/2}$

$\psi = (C\nu x^{m+1})^{1/2} f[y(C/\nu)^{1/2} x^{(m-1)/2}]$

$$v = -\frac{\partial\psi}{\partial x} = -\frac{m+1}{2} x^{(m-1)/2} (C\nu)^{1/2} f(\eta) - (C\nu x^{m+1})^{1/2} \frac{df}{d\eta} y(C/\nu)^{1/2} \frac{m-1}{2} x^{(m-3)/2}$$

$f'(0) = 0$

$f'(\infty) = 1$

At the wall, the normal velocity $v_0 = v(y = 0)$ ➡ $v_0 = -\dfrac{m+1}{2} x^{(m-1)/2} (C\nu)^{1/2} f(0)$

if $f(0) =$ constant ➡ v_0 must vary as $x^{(m-1)/2}$

$$f(0) = -\frac{2}{m+1} \frac{v_0}{U_\infty} \mathrm{Re}_x^{1/2} \quad \text{(constant)}$$

$\mathrm{Re}_x = U_\infty x/\nu$

$(v_0/U_\infty)\ \mathrm{Re}_x^{1/2}$ is the *blowing parameter*

Table 2.4 Effect of flow through the wall: local skin friction coefficient and Nusselt number for laminar boundary layer flow over a permeable isothermal wall parallel to the stream

$\frac{v_0}{U_\infty}\mathrm{Re}_x^{1/2}$	$f''(0) = \frac{1}{2}C_{f,x}\ \mathrm{Re}_x^{1/2}$	$\mathrm{Nu}/\mathrm{Re}_x^{1/2}$			
		$\mathrm{Pr} = 0.7$	$\mathrm{Pr} = 0.8$	$\mathrm{Pr} = 0.9$	
−2.5	2.59	1.85	2.097	2.59	Suction
−0.75	0.945	0.722	0.797	0.945	Suction
−0.25	0.523	0.429	0.461	0.523	Suction
0	0.332	0.292	0.307	0.332	Impermeable wall
+0.25	0.165	0.166	0.166	0.165	Blowing
+0.375	0.094	0.107	0.103	0.0937	Blowing
+0.5	0.036	0.0517	0.0458	0.0356	Blowing
+0.619	0	0	0	0	Separation

when the wall is parallel to the free stream

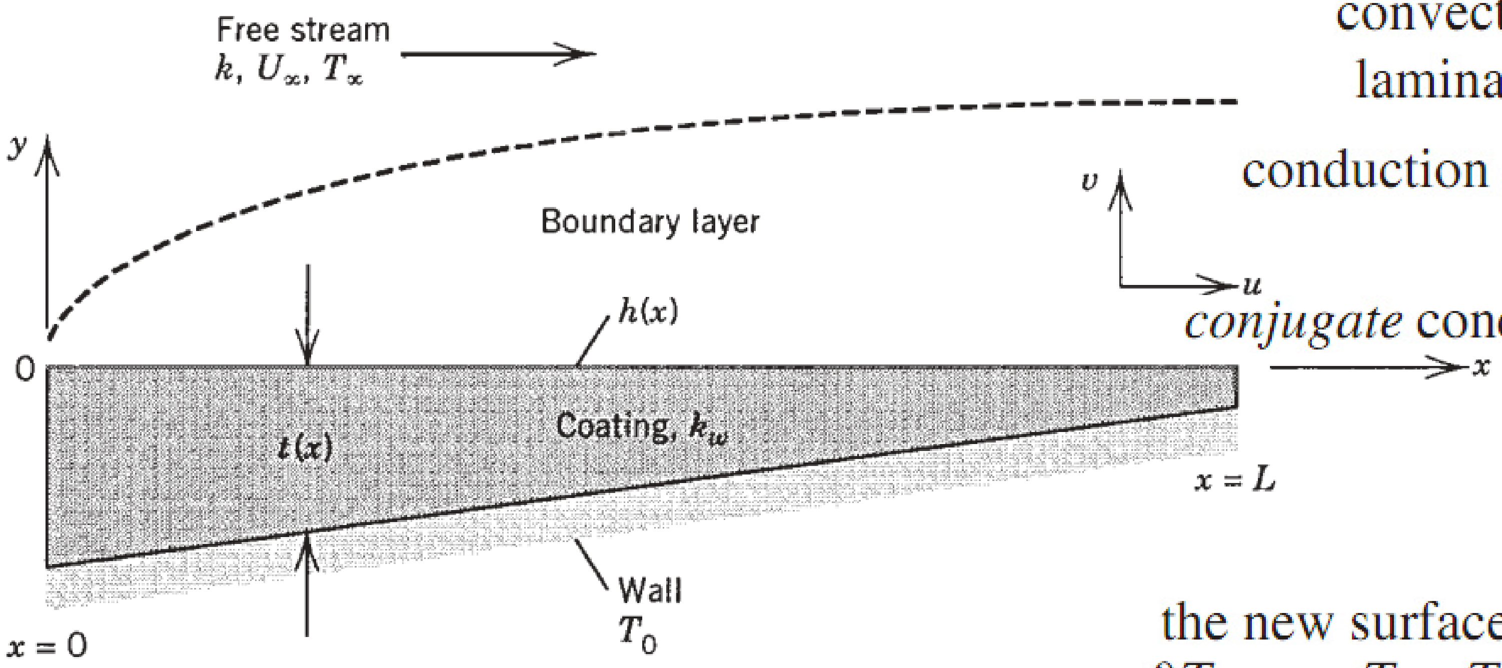
Table 2.5 Local Nusselt number $Nu/Re_x^{1/2}$ for laminar boundary layer flow over an isothermal wedge with blowing (Pr = 0.7)

$\frac{v_0}{U_\infty}Re_x^{1/2}$	*m*							
	−0.0418 ($\beta/\pi = -0.08$)	−0.0036 (−0.0072)	0 (0)	0.0257 (0.05)	0.0811 (0.15)	0.333 (1/2)	0.500 (2/3)	1 (1)
0			0.292			0.384		0.496
0.0239	0.103							
0.25			0.166					
0.333						0.242		
0.375			0.107				0.259	
0.5		0.0251	0.0517					0.293
0.518				0.087				
0.558					0.109			
0.667						0.131		
1								0.146

Table 2.6 Effect of blowing on the local Nusselt number in laminar stagnation flow on an isothermal axisymmetric body (Pr = 0.7)

$\frac{v_0}{U_\infty}\mathrm{Re}_x^{1/2}$	0	0.567	1.154
$\mathrm{Nu}/\mathrm{Re}_x^{1/2}$	0.664	0.419	0.227

CONDUCTION ACROSS A SOLID COATING DEPOSITED ONAWALL



convection across the laminar boundary layer

conduction across the $t(x)$ layer

conjugate conduction and convection,

the new surface condition

$$k\frac{\partial T}{\partial y} = k_w \frac{T - T_0}{t} \qquad (y = 0)$$

By: M. Farhadi, Faculty of Mechanical Engineering, Babol University of Technology

$$J\frac{\partial\theta}{\partial\eta} = \theta \qquad (\eta = 0) \qquad \text{where } \theta = (T - T_0)/(T_\infty - T_0),\ \eta = y(U_\infty/\nu x)^{1/2}, \text{ and}$$

$$J = \frac{k}{k_w}\left(\frac{U_\infty t^2}{\nu x}\right)^{1/2}$$

$J = 0$ corresponds to the Pohlhausen problem

$$t(x) = \bar{t}\left[1 + b\left(\frac{1}{2} - \frac{x}{L}\right)\right]$$

$\bar{t}$ is the coating thickness averaged from $x = 0$ to $x = L$,

b is a dimensionless taper parameter.

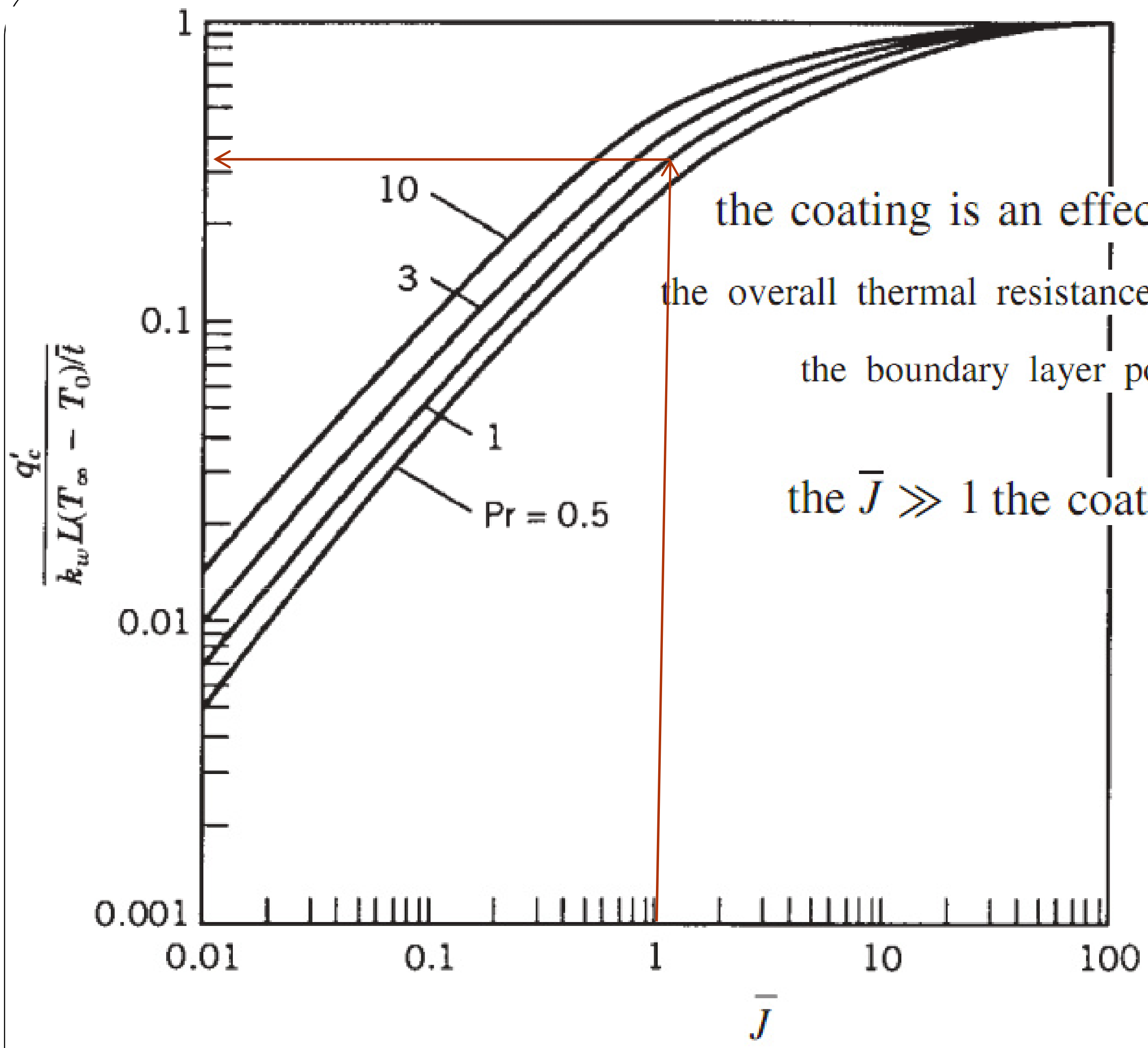
$$J = \bar{J}\left(\frac{x}{L}\right)^{-1/2}\left[1 + b\left(\frac{1}{2} - \frac{x}{L}\right)\right]$$

in which $\bar{J}$ is the J value based on the L-averaged thickness $\bar{t}$,

$$\bar{J} = \frac{k}{k_w}\frac{\bar{t}}{L}\mathrm{Re}_L^{1/2}$$

$$q' = \int_0^L k\left(\frac{\partial T}{\partial y}\right)_{y=0} dx = k(T_\infty - T_0)\mathrm{Re}_L^{1/2}\int_0^1 \theta'(0)\left(\frac{x}{L}\right)^{-1/2} d\left(\frac{x}{L}\right)$$

a single similarity solution for θ does not exist in this case,



the coating is an effective insulator, $k_w L(T_\infty - T_0)/\bar{t}$.

the overall thermal resistance is dominated by the coating ($\bar{J} \gg 1$)

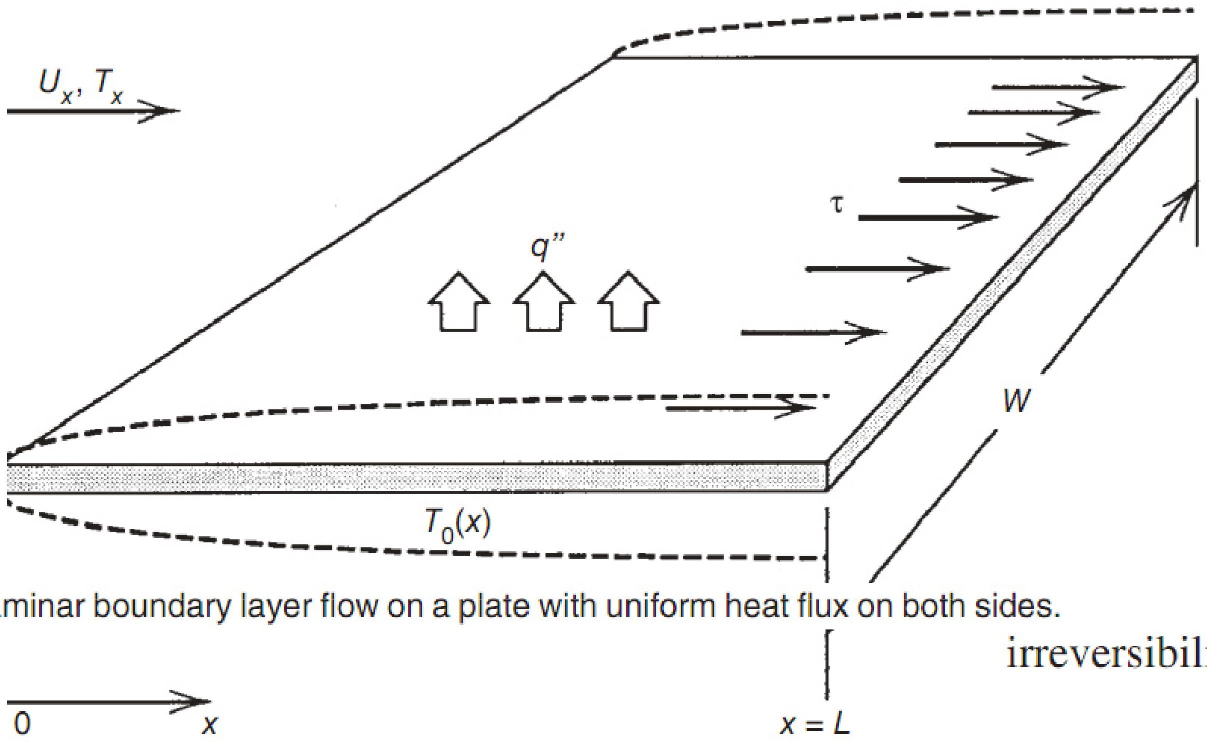
the boundary layer poses the greater resistance ($\bar{J} \ll 1$)

the $\bar{J} \gg 1$ the coating is an effective insulator

$$k_w L(T_\infty - T_0)/\bar{t}$$

Figure 2.12 Total heat transfer rate when the wall of Fig. 12.11 is coated with a layer of uniform thickness ($b = 0$).

ENTROPY GENERATION MINIMIZATION IN LAMINAR BOUNDARY LAYER FLOW



Laminar boundary layer flow on a plate with uniform heat flux on both sides.

$T_0 > T_\infty$

$q''(T_0 - T_\infty)$ is always positive

A fundamental result in thermodynamics is

irreversibility of heat transfer

$$S_{\text{gen}} = \frac{1}{T_\infty^2}\int_A q''(T_0 - T_\infty)\, dA + \frac{F_D U_\infty}{T_\infty}$$

irreversibility of fluid flow

A is the body surface area.

T_0 is the surface temperature

F_D is the drag force

$T_0 - T_\infty$ is small relative to the absolute temperature T_∞.

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$$\int_A q''(T_0 - T_\infty)\, dA = q''(\overline{T}_0 - T_\infty)(2LW)$$

$$\text{Nu} = \frac{q''}{T_0(x) - T_\infty}\frac{x}{k} = 0.453\text{Pr}^{1/3}\text{Re}_x^{1/2} \qquad (0.5 < \text{Pr} < 10)$$

Uniform Heat Flux

$$\frac{0.736(q')^2}{T_\infty^2 k \ \text{Pr}^{1/3}\text{Re}_L^{1/2}}$$

$$q' = 2Lq''.$$

$$F_D = 2LW\tau_{0-L}.$$

$$\tau_{0-L} = 0.664\rho U_\infty^2 \text{Re}_L^{-1/2}$$

$$\frac{F_D U_\infty}{T_\infty} = 1.328\frac{\mu}{T_\infty}U_\infty^2\text{Re}_L^{1/2}$$

$$\frac{S_{\text{gen}}}{W} = \frac{0.736(q')^2}{T_\infty^2 k \ \text{Pr}^{1/3}\text{Re}_L^{1/2}} + 1.328\frac{\mu}{T_\infty}U_\infty^2\text{Re}_L^{1/2}$$

$$\text{Re}_L = U_\infty L/\nu$$

irreversibility due to heat transfer decreases as the plate is made longer, while the fluid flow irreversibility increases

S_{gen} is minimum when L has a certain value

$$\partial S_{\text{gen}}/\partial \text{Re}_L = 0,$$

$$\text{Re}_{L,\text{opt}} = 0.554B^2$$

$$B = \frac{q'/U_\infty}{(k\mu T_\infty \ \text{Pr}^{1/3})^{1/2}}$$

$$\text{Re}_{L,\text{opt}} = U_\infty L_{\text{opt}}/\nu$$

The number B is the dimensionless version of the ratio of the heat transfer rate divided by the flow speed,

$\mathrm{Re}_L \ll B^2$ ⟹ the entropy generation rate is due mainly to heat transfer

$\mathrm{Re}_L \gg B^2$ ⟹ the plate is so long that most of its work destruction is due to fluid friction.

its swept length for minimum irreversibility

$$L_{\mathrm{opt}} = 0.554 \frac{(q')^2}{k T_\infty \rho U_\infty^3 \ \mathrm{Pr}^{1/3}}$$

$$S_{\mathrm{gen,min}} = 1.98 \frac{q U_\infty}{(k/\mu)^{1/2} T_\infty^{3/2} \ \mathrm{Pr}^{1/6}}$$

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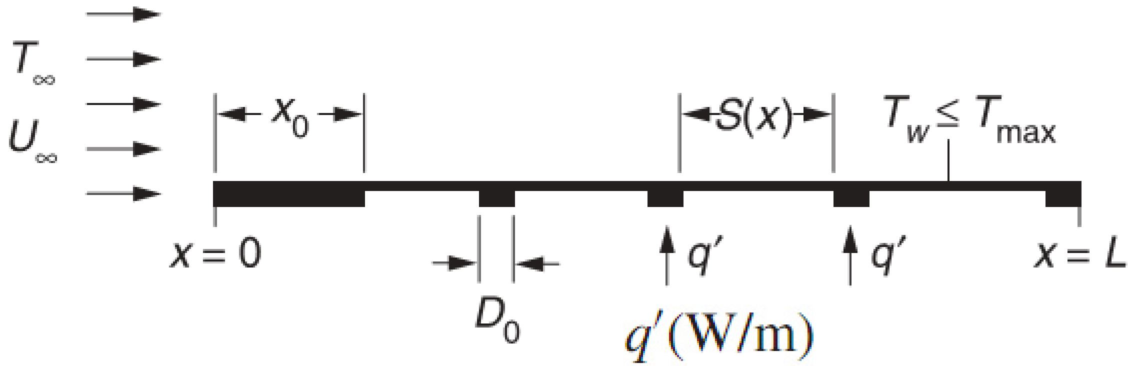
2.12 DISTRIBUTION OF HEAT SOURCES ON A WALL COOLED BY FORCED CONVECTION



the wall temperature is near the allowed limit

$$T_w(x) = T_{max}, \text{ constant}$$

The number of heat sources per unit of plate length (N') is unknown.

First, we assume that the density of line sources is sufficiently high so that we may express the distribution of discrete q’ sources as a nearly continuous distribution of heat flux

$$\mathrm{Re} = U_\infty L/\nu$$

$$q''(x) = q'N'$$

$$\mathrm{Pr} \gtrsim 1$$

با توجه با ثابت بودن دمای سطح

$$\mathrm{Nu} = 0.332\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}$$

$$\mathrm{Nu} = \frac{hx}{k}$$

$$\frac{q''x}{k\,(T_{max} - T_\infty)} = 0.332\,\mathrm{Pr}^{1/3}\left(\frac{U_\infty x}{\nu}\right)^{1/2}$$

$$N'(x) = 0.332\,\frac{k}{q'}\,(T_{max} - T_\infty)\,\mathrm{Pr}^{1/3}\left(\frac{U_\infty}{\nu}\right)^{1/2} x^{-1/2}$$

$$N = \int_0^L N'\,dx = 0.664\,\frac{k}{q'}\,(T_{max} - T_\infty)\,\mathrm{Pr}^{1/3}\,\mathrm{Re}^{1/2}$$

$$Q'_{max} = q'N = 0.664k(T_{max} - T_{\infty}) \, Pr^{1/3} \, Re^{1/2}$$

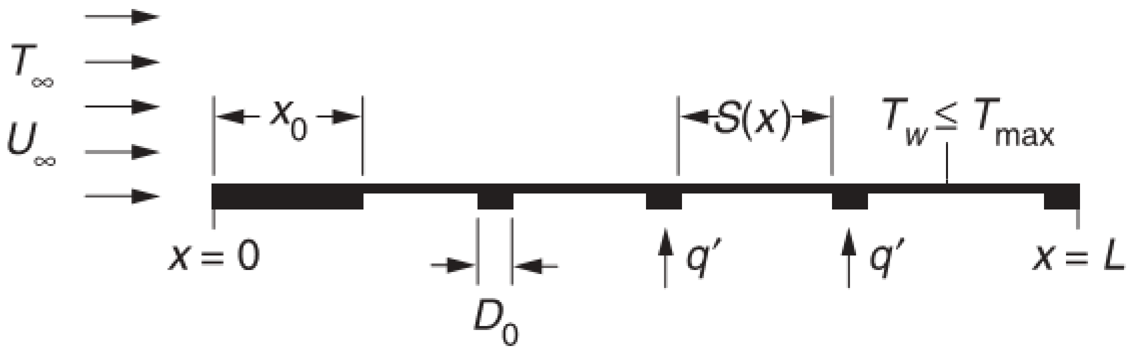
This Q'_{max} expression is the same as the total heat transfer rate from an isothermal wall at T_{max}

the local number of heat sources per unit of wall height is

طول یک متر

$$N'(x) = \frac{1}{D_0 + S(x)}$$

فاصله قرار گرفتن هر هیتر



$$(q''_0 = q'/D_0)$$

The heat flux q''_0 is a known constant ➡ the function $q'(x)$

$$N'(x) = 0.332 \frac{k}{q'} (T_{max} - T_{\infty}) \, Pr^{1/3} \left(\frac{U_{\infty}}{\nu}\right)^{1/2} x^{-1/2}$$

$$N'(x) = \frac{1}{D_0 + S(x)}$$

$$\frac{S(x)}{L} \cong \frac{3q' \, Pr^{-1/3} \, Re^{-1/2}}{k(T_{max} - T_{\infty})} \left(\frac{x}{L}\right)^{1/2} - \frac{D_0}{L}$$

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$$\frac{S(x)}{L} \cong \frac{3q' \, \mathrm{Pr}^{-1/3} \, \mathrm{Re}^{-1/2}}{k(T_{max} - T_\infty)} \left(\frac{x}{L}\right)^{1/2} - \frac{D_0}{L}$$

The spacing S increases as x increases

Near the start of the boundary layer, the $S(x)$ has negative values.

Because D_0 is the smallest length scale of the structure, the spacings S cannot be smaller than D_0

$S \sim D_0$ when $x \sim x_0$

$$\frac{S(x)}{L} \cong \frac{3q' \, \mathrm{Pr}^{-1/3} \, \mathrm{Re}^{-1/2}}{k(T_{max} - T_\infty)} \left(\frac{x}{L}\right)^{1/2} - \frac{D_0}{L}$$

$$\left(\frac{x_0}{L}\right)^{1/2} \sim 0.664 \, \frac{D_0}{L} \, \frac{k}{q'} \, (T_{max} - T_0) \, \mathrm{Pr}^{1/3} \, \mathrm{Re}^{1/2}$$

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