

Table 6.1 Summary of physical observations concerning the beginning of transition from laminar to turbulent flow

Flow Configuration	Condition ^a Necessary for the Existence of Laminar Flow	Source	Observations
Boundary layer flow (without longitudinal pressure gradient)	$Re < 3.5 \times 10^5$ $Re < 2 \times 10^4 - 10^6$	[3] ^b	Re is the Reynolds number based on wall length and free-stream velocity
Duct flow	$Re < 2000$		Re is based on hydraulic diameter and duct-averaged velocity
Free-jet flow (axisymmetric)	$Re < 10 - 30$	[4]	Re is based on nozzle diameter and mean velocity through the nozzle
Wake flow (two-dimensional)	$Re < 32$	[5]	Re is based on the cylinder diameter and free-stream velocity

Natural convection

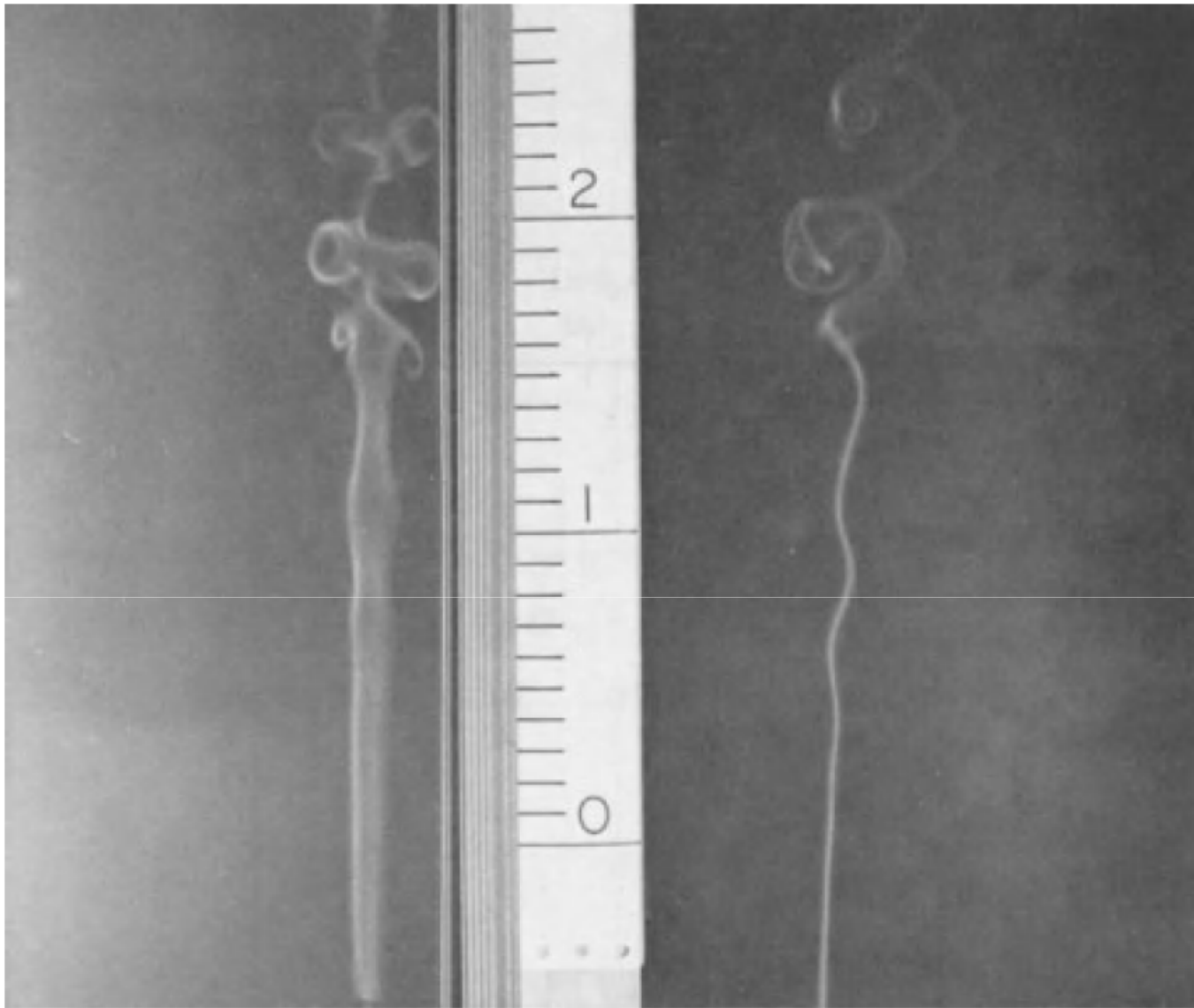
boundary layer flow

Isothermal wall	$Gr < 1.5 \times 10^9$ (Pr = 0.71) [6] ^c $Gr < 1.3 \times 10^9$ (Pr = 6.7) [6, 7]	The Grashof number Gr is based on wall height and wall–ambient temperature difference
Constant-heat-flux wall	$Gr_* < 1.6 \times 10^{10}$ (Pr = 0.71) [6] $Gr_* < 6.6 \times 10^{10}$ (Pr = 6.7) [8]	Gr_* is the Grashof number based on heat flux and wall height [see eq. (4.70), where $Gr_* = Ra_*/Pr$]
Plume flow (axisymmetric)	$Ra_q < 10^{10}$ (Pr = 0.71) [9]	Ra_q is the Rayleigh number based on heat source strength and plume height [see eq. (6.6)]
Film condensation on a vertical plate	$\frac{4\Gamma}{\mu} < 1800$ [10]	Γ is the condensate mass flow rate per unit of film width

^aAll numerical values are order-of-magnitude approximate and vary from one experimental report to another.

^bThe transition is triggered by velocity disturbances in excess of 18 percent of the free-stream velocity.

^cAveraged from the data compiled in Ref. 6.



$$\lambda_B \sim y_{tr}$$

a sinusoidal shape of characteristic wavelength λ_B

TURBULENT BOUNDARY LAYER FLOW



Figure 7.1 Turbulent boundary layer in boiling water flowing from left to right over a flat surface; $U_\infty = 0.52 \text{ m/s}$, $q''_0 = 4.8 \times 10^5 \text{ W/m}^2$. (Reprinted with permission from J. H. Lienhard, *A Heat Transfer Textbook*, 1981, p. 417. Copyright © 1981 Prentice-Hall, Inc.)

TIME-AVERAGED EQUATIONS

mean values

$$u = \bar{u} + u', \quad P = \bar{P} + P'$$

$$v = \bar{v} + v', \quad T = \bar{T} + T'$$

$$w = \bar{w} + w'$$

$$\bar{u} = \frac{1}{\text{period}} \int_0^{\text{period}} u \, d(\text{time})$$

$$\int_0^{\text{period}} u' \, d(\text{time}) = 0$$

fluctuating

$$\begin{aligned} \overline{u+v} &= \bar{u} + \bar{v} & \overline{uv} &= \bar{u}\bar{v} + \overline{u'v'} & \overline{\frac{\partial u}{\partial x}} &= \frac{\partial \bar{u}}{\partial x} & \overline{\frac{\partial u}{\partial t}} &= 0 & \overline{\frac{\partial u}{\partial t}} &= 0 \\ \overline{uu'} &= 0 & \overline{u^2} &= \bar{u}^2 + \overline{u'^2} & \overline{\frac{\partial u}{\partial x}} &= \frac{\partial \bar{u}}{\partial x} & \overline{\frac{\partial u}{\partial t}} &= 0 & \overline{\frac{\partial u}{\partial t}} &= 0 \end{aligned}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \rightarrow \quad \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{u}'}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{v}'}{\partial y} + \frac{\partial \bar{w}}{\partial z} + \frac{\partial \bar{w}'}{\partial z} = 0$$

$$\boxed{\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0}$$

the x momentum equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw) = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \nabla^2 u$$

Averaging each term over time

$$\frac{\partial}{\partial x}(\overline{u^2}) + \frac{\partial}{\partial y}(\overline{uv}) + \frac{\partial}{\partial z}(\overline{uw}) = -\frac{1}{\rho} \frac{\partial \bar{P}}{\partial x} + \nu \nabla^2 \bar{u}$$

$$\frac{\partial}{\partial x}(\overline{u^2}) + \frac{\partial}{\partial y}(\overline{uv}) + \frac{\partial}{\partial z}(\overline{uw}) = -\frac{1}{\rho} \frac{\partial \bar{P}}{\partial x} + \nu \nabla^2 \bar{u} - \frac{\partial}{\partial x}(\overline{u'^2}) - \frac{\partial}{\partial y}(\overline{u'v'}) - \frac{\partial}{\partial z}(\overline{u'w'})$$

$$\begin{aligned} \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} = & - \frac{1}{\rho} \frac{\partial \bar{P}}{\partial x} + \nu \nabla^2 \bar{u} - \frac{\partial}{\partial x} (\overline{u'^2}) \\ & - \frac{\partial}{\partial y} (\overline{u'v'}) - \frac{\partial}{\partial z} (\overline{u'w'}) \end{aligned}$$

time-averaged of the momentum equations in the y and z directions are

$$\begin{aligned} \bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} + \bar{w} \frac{\partial \bar{v}}{\partial z} = & - \frac{1}{\rho} \frac{\partial \bar{P}}{\partial y} + \nu \nabla^2 \bar{v} - \frac{\partial}{\partial x} (\overline{u'v'}) \\ & - \frac{\partial}{\partial y} (\overline{v'^2}) - \frac{\partial}{\partial z} (\overline{v'w'}) \end{aligned}$$

$$\begin{aligned} \bar{u} \frac{\partial \bar{w}}{\partial x} + \bar{v} \frac{\partial \bar{w}}{\partial y} + \bar{w} \frac{\partial \bar{w}}{\partial z} = & - \frac{1}{\rho} \frac{\partial \bar{P}}{\partial z} + \nu \nabla^2 \bar{w} - \frac{\partial}{\partial x} (\overline{u'w'}) \\ & - \frac{\partial}{\partial y} (\overline{v'w'}) - \frac{\partial}{\partial z} (\overline{w'^2}) \end{aligned}$$

the energy equation

$$\bar{u} \frac{\partial \bar{T}}{\partial x} + \bar{v} \frac{\partial \bar{T}}{\partial y} + \bar{w} \frac{\partial \bar{T}}{\partial z} = \alpha \nabla^2 \bar{T} - \frac{\partial}{\partial x} (\overline{u'T'}) - \frac{\partial}{\partial y} (\overline{v'T'}) - \frac{\partial}{\partial z} (\overline{w'T'})$$

17 unknowns

the unknowns are \bar{u} , \bar{v} , \bar{w} , \bar{P} , \bar{T} , and the 12 terms of type of fluctuating quantities

hence, the *closure problem*

BOUNDARY LAYER EQUATIONS

in the boundary layer, we can neglect $(\partial/\partial x)(\overline{u'^2})$ relative to $(\partial/\partial y)(\overline{u'v'})$

$(\partial/\partial x)(\overline{u'T'})$ relative to $(\partial/\partial y)(\overline{v'T'})$

Applying the other simplifications

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho} \frac{d\bar{P}}{dx} + \nu \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial}{\partial y} (\overline{u'v'})$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

$$\bar{u} \frac{\partial \bar{T}}{\partial x} + \bar{v} \frac{\partial \bar{T}}{\partial y} = \alpha \frac{\partial^2 \bar{T}}{\partial y^2} - \frac{\partial}{\partial y} (\overline{v'T'})$$

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho} \frac{d\bar{P}}{dx} + \nu \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial}{\partial y} (\overline{u'v'})$$

$$\bar{u} \frac{\partial \bar{T}}{\partial x} + \bar{v} \frac{\partial \bar{T}}{\partial y} = \alpha \frac{\partial^2 \bar{T}}{\partial y^2} - \frac{\partial}{\partial y} (\overline{v'T'})$$

to rewrite eqs.



$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho} \frac{d\bar{P}}{dx} + \frac{1}{\rho} \frac{\partial}{\partial y} \left(\mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'} \right)$$

$$\bar{u} \frac{\partial \bar{T}}{\partial x} + \bar{v} \frac{\partial \bar{T}}{\partial y} = \frac{1}{\rho c_P} \frac{\partial}{\partial y} \left(k \frac{\partial \bar{T}}{\partial y} - \rho c_P \overline{v'T'} \right)$$

the products $\overline{u'v'}$ and $\overline{v'T'}$ \rightarrow they are nonzero, are they negative or positive?

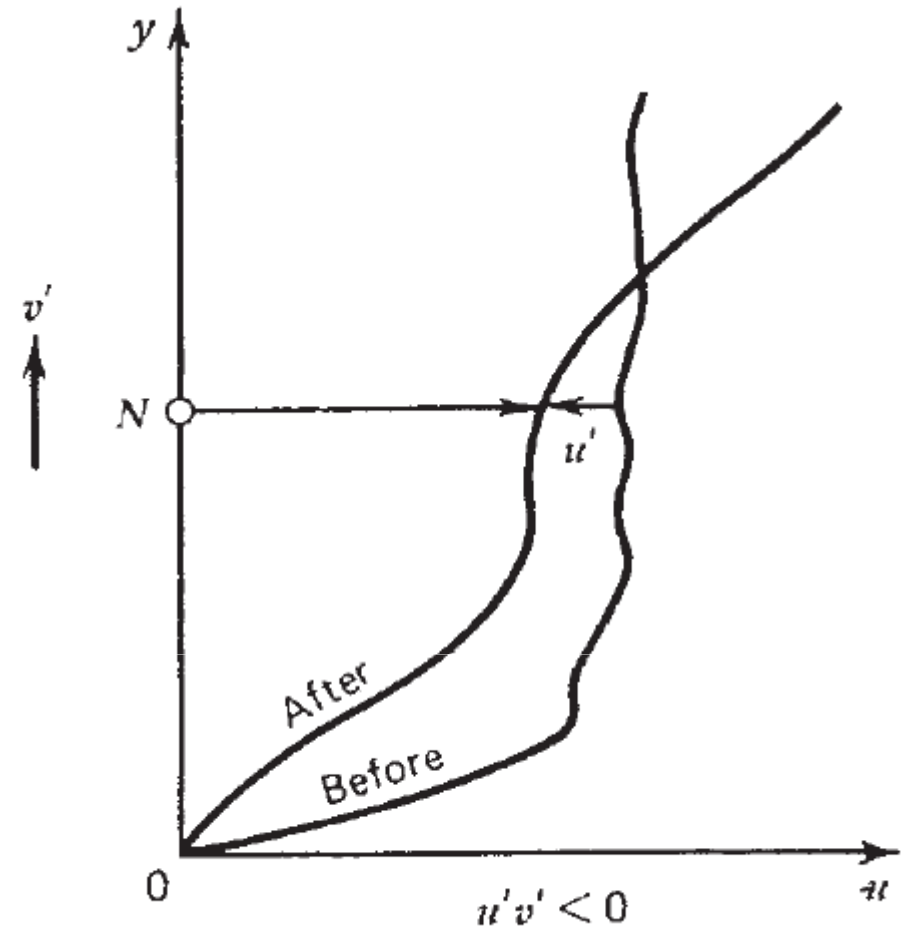
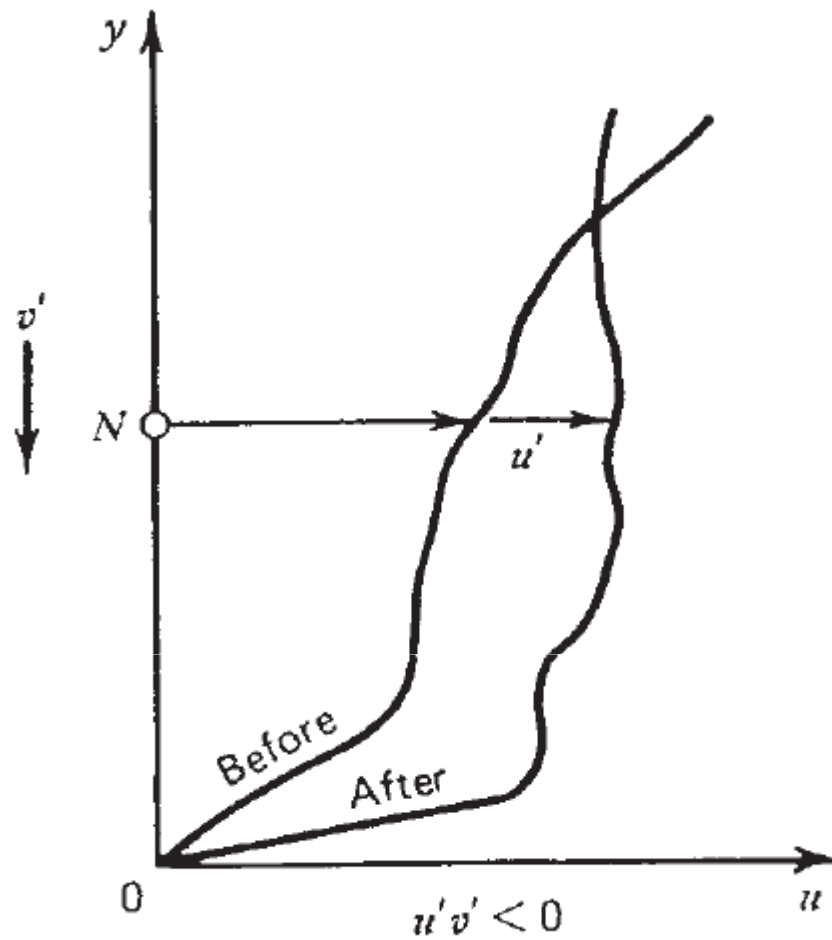
the product $u'v'$ emerges as a negative

$$-\rho \overline{u'v'} = \rho \epsilon_M \frac{\partial \bar{u}}{\partial y}$$

eddy shear stress

$$-\rho c_P \overline{v'T'} = \rho c_P \epsilon_H \frac{\partial \bar{T}}{\partial y}$$

eddy heat flux



$v' < 0$; increase the longitudinal velocity to induce a positive fluctuation $u' > 0$.
the product $u'v'$ emerges as a negative

$$\tau_{\text{app}} = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'} = \rho(\nu + \epsilon_M) \frac{\partial \bar{u}}{\partial y} \quad \text{apparent shear stress}$$

$$-q''_{\text{app}} = k \frac{\partial \bar{T}}{\partial y} - \rho c_P \overline{v'T'} = \rho c_P(\alpha + \epsilon_H) \frac{\partial \bar{T}}{\partial y} \quad \text{apparent heat flux}$$

into the boundary layer equations

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho} \frac{d\bar{P}}{dx} + \frac{\partial}{\partial y} \left[(\nu + \epsilon_M) \frac{\partial \bar{u}}{\partial y} \right]$$

$$\bar{u} \frac{\partial \bar{T}}{\partial x} + \bar{v} \frac{\partial \bar{T}}{\partial y} = \frac{\partial}{\partial y} \left[(\alpha + \epsilon_H) \frac{\partial \bar{T}}{\partial y} \right]$$

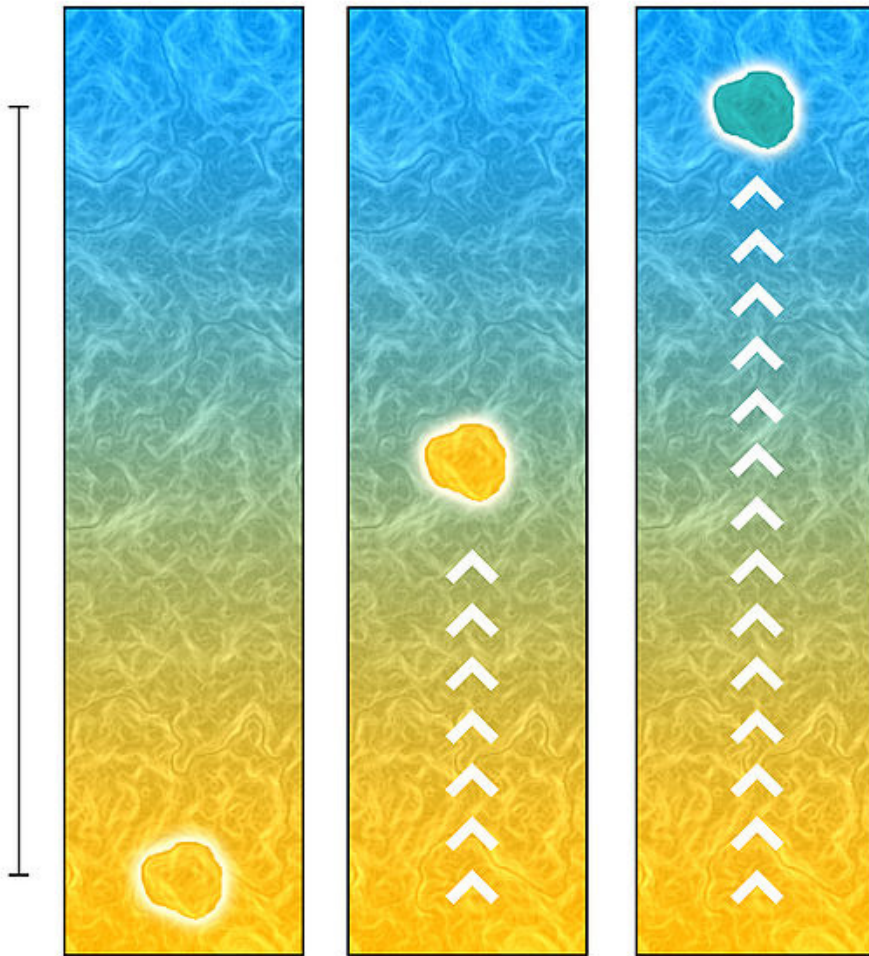
where ϵ_M and ϵ_H are two empirical functions known as

momentum eddy diffusivity and *thermal eddy diffusivity*,

Note that ϵ_M and ϵ_H are *flow parameters*, not fluid properties.

five unknowns (\bar{u} , \bar{v} , \bar{T} , ϵ_M , and ϵ_H).

MIXING LENGTH MODEL



a length called mixing length which is the average distance perpendicular to flow a small fluid mass will travel before its momentum is changed by new environment.

Fluid, which comes to the layer y_1 from a layer $(y_1 - l)$ has a positive value of v' . If the lump of fluid retains its original momentum then its velocity at its current location y_1 is smaller than the velocity prevailing there. The difference in velocities is then

$$\Delta u_1 = \bar{u}(y_1) - \bar{u}(y_1 - l) \approx l \left(\frac{\partial \bar{u}}{\partial y} \right)_{y_1}$$

$$\Delta u_2 = \bar{u}(y_1 + l) - \bar{u}(y_1) \approx l \left(\frac{\partial \bar{u}}{\partial y} \right)_{y_1}$$

$$|\bar{u}'| = \frac{1}{2} (|\Delta u_1| + |\Delta u_2|) = l \left| \left(\frac{\partial \bar{u}}{\partial y} \right) \right|_{y_1}$$

$$\overline{u'v'} = -C_1 |\bar{u}'| |\bar{v}'|$$

$$\overline{u'v'} = -C_2 l^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2$$

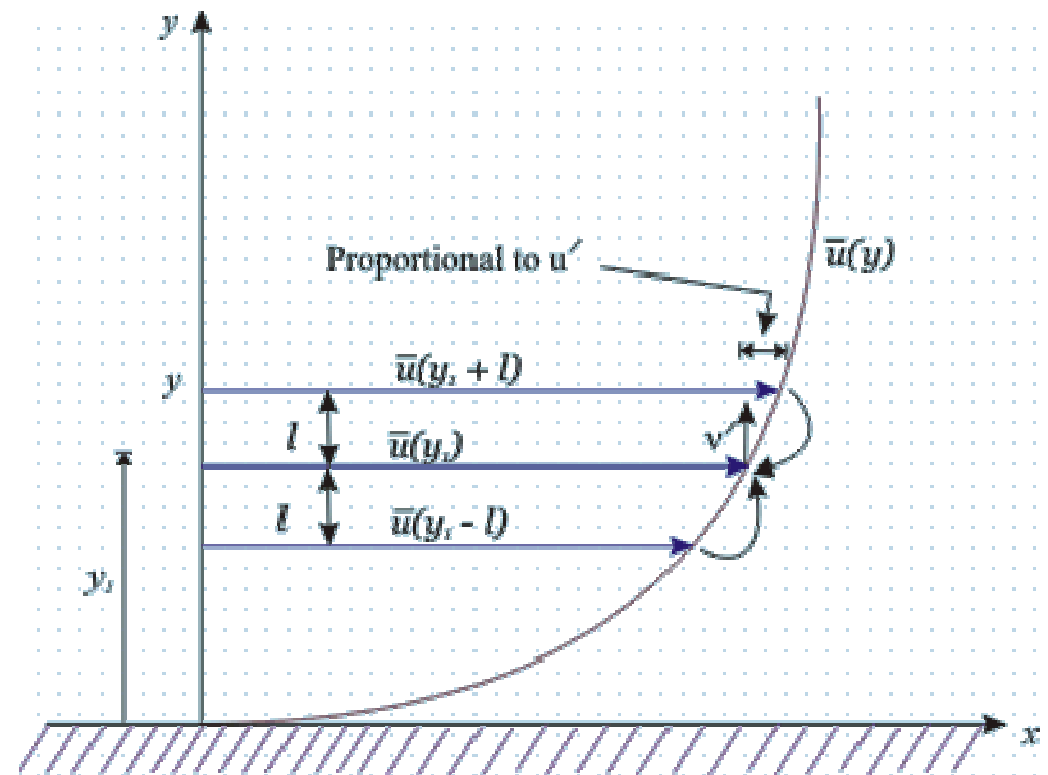
$$\tau_t = -\rho \overline{u'v'} = \mu_t \frac{\partial \bar{u}}{\partial y}$$

$$\mu_t = \rho l^2 \left| \frac{\partial \bar{u}}{\partial y} \right|$$

$$\nu_t = l^2 \left| \frac{\partial \bar{u}}{\partial y} \right|$$

$$|\bar{v}'| \sim |\bar{u}'|$$

$$\text{or, } |\bar{v}'| = (\text{const}) |\bar{u}'| = (\text{const}) l \left| \left(\frac{\partial \bar{u}}{\partial y} \right) \right|$$



$$\epsilon_M = l^2 \left| \frac{\partial \bar{u}}{\partial y} \right|$$

$$l = \kappa y$$

κ is an empirical constant
for von Kármán's constant κ turns out to be 0.4

$$\epsilon_M = \kappa^2 y^2 \left| \frac{\partial \bar{u}}{\partial y} \right|$$

uniform flow parallel to a flat wall.

apparent shear stress $(\nu + \epsilon_M)(\partial \bar{u}/\partial y)$ that does not vary with y

$$(\nu + \epsilon_M) \frac{\partial \bar{u}}{\partial y} = \frac{\tau_0}{\rho} \quad \text{at } y = 0 \rightarrow \text{the Reynolds stress } -\rho \overline{u'v'} \text{ vanishes.}$$

$$\text{at } y = 0 \rightarrow (\nu + \cancel{\epsilon_M}) \frac{\partial \bar{u}}{\partial y} \rightarrow \tau_0 \text{ is the actual wall shear stress}$$

the dimensions of $(\tau_0/\rho)^{1/2}$ = velocity

friction velocity

$$u_* = \left(\frac{\tau_0}{\rho} \right)^{1/2}$$

nondimensionalization of the flow problem

$$u^+ = \frac{\bar{u}}{u_*}, \quad v^+ = \frac{\bar{v}}{u_*}$$

$$x^+ = \frac{xu_*}{\nu}, \quad y^+ = \frac{yu_*}{\nu}$$

$$\left. \begin{aligned}
 u^+ &= \frac{\bar{u}}{u_*}, \\
 y^+ &= \frac{yu_*}{\nu} \\
 u_* &= \left(\frac{\tau_0}{\rho} \right)^{1/2}
 \end{aligned} \right\} \left(1 + \frac{\epsilon_M}{\nu} \right) \frac{du^+}{dy^+} = 1$$

depending on the relative size of ϵ_M and ν .
 ϵ_M/ν must vary with the distance measured away from the wall

1. The *viscous sublayer* (VSL), where $\nu \gg \epsilon_M$
2. The *fully turbulent sublayer* (or the *turbulent core*), where $\epsilon_M \gg \nu$

$$\left. \begin{aligned}
 y_{\text{VSL}}^+ &\rightarrow \text{Neglecting the term } \epsilon_M/\nu \\
 u^+(0) &= 0,
 \end{aligned} \right\} u^+ = y^+$$

In the fully turbulent sublayer

$$\left(1 + \frac{\epsilon_M}{\nu} \right) \frac{du^+}{dy^+} = 1 \rightarrow \frac{\epsilon_M}{\nu} \frac{du^+}{dy^+} = 1 \left\{ \begin{aligned}
 &\kappa^2 (y^+)^2 \left(\frac{du^+}{dy^+} \right)^2 = 1 \\
 &\epsilon_M = \kappa^2 y^2 \left| \frac{\partial \bar{u}}{\partial y} \right|
 \end{aligned} \right.$$

Integrating this equation from the sublayer interface y_{VSL}^+ to any y^+

$$\int \kappa^2 (y^+)^2 \left(\frac{du^+}{dy^+} \right)^2 = 1 \quad \rightarrow \quad u^+ = \frac{1}{\kappa} \ln y^+ + y_{\text{VSL}}^+ - \frac{1}{\kappa} \ln y_{\text{VSL}}^+$$

$$u^+ = A \ln y^+ + B$$

where A and B are two empirical constants.

experimental measurements $\rightarrow A \cong 2.5$ and $B \cong 5.5$

$$\kappa \cong 0.4 \quad \text{and} \quad y_{\text{VSL}}^+ \cong 11.6$$

$y^+ = y_{\text{VSL}}^+$, where neither ν nor ϵ_M can be neglected

The experimental observation that the transition from the viscous sublayer to the fully turbulent sublayer takes place around $y_{\text{VSL}}^+ = O(10)$

The laminar shear layer grows until the local Reynolds number based on local thickness becomes of order 10^2

$$\frac{y_{\text{VSL}} U_\infty}{\nu} \sim 10^2$$

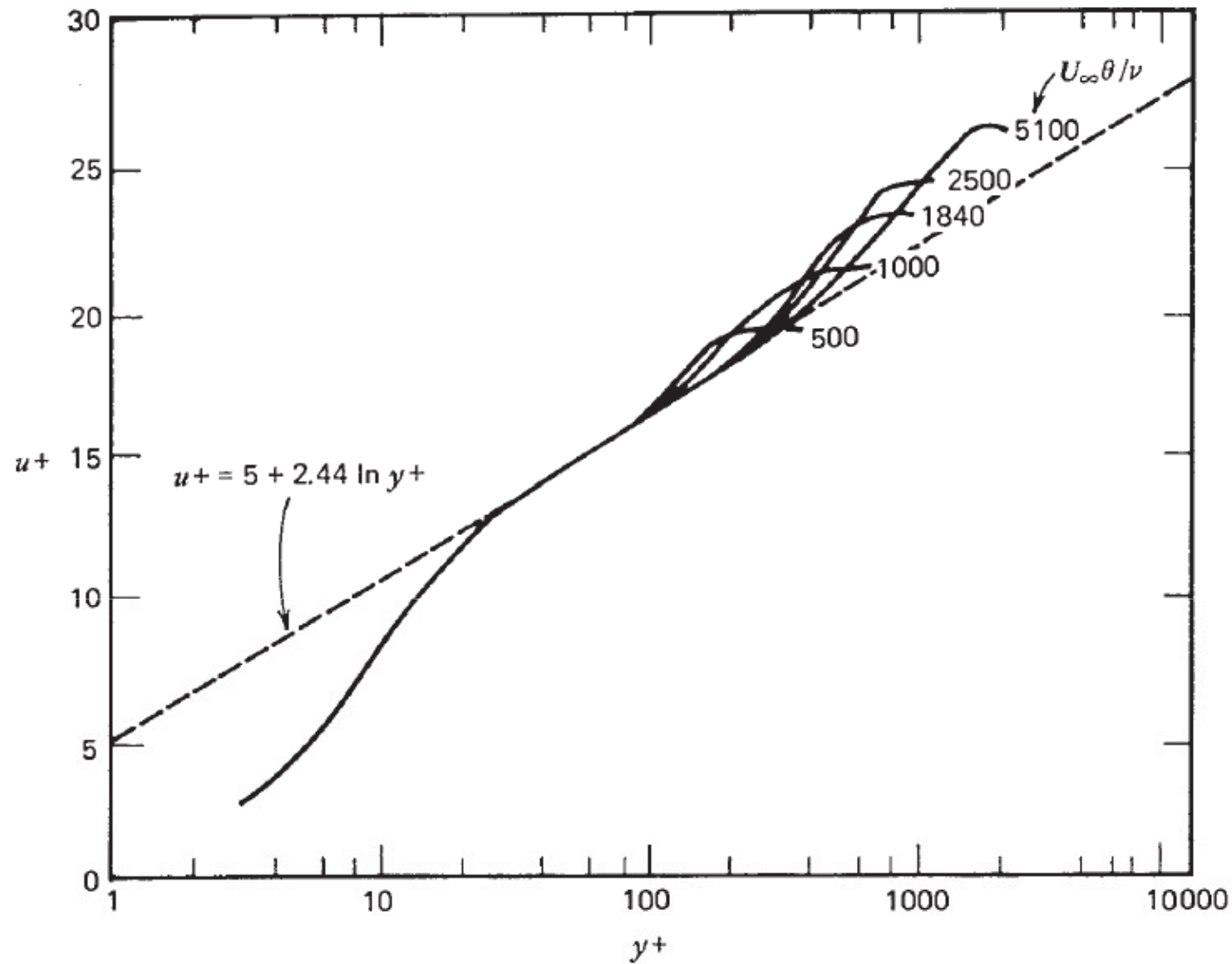


Figure 7.4 Example of $u^+(y^+)$ velocity measurements in turbulent boundary layer flow without longitudinal pressure gradient. Note the use of $A \cong 2.44$ and $B \cong 5$ to fit the data. (Reprinted with permission from L. P. Purtell et al., *Physics of Fluids*, Vol. 24, pp. 802–811, May 1981. Copyright © 1981 American Institute of Physics.)

Table 7.1 Summary of longitudinal velocity expressions for the inner region of a turbulent boundary layer

$u^+(y^+)$	Range	Reference
$u^+ = y^+$ $u^+ = 2.5 \ln y^+ + 5.5$	$0 < y^+ < 11.6$ $y^+ > 11.6$	Prandtl and Taylor [9]
$u^+ = y^+$ $u^+ = 5 \ln y^+ - 3.05$ $u^+ = 2.5 \ln y^+ + 5.5$	$0 < y^+ < 5$ $5 < y^+ < 30$ $y^+ > 30$	von Kármán [12]
$u^+ = 14.53 \tanh(y^+/14.53)$ $u^+ = 2.5 \ln y^+ + 5.5$	$0 < y^+ < 27.5$ $y^+ > 27.5$	Rannie [13]
$\frac{du^+}{dy^+} = \frac{2}{1 + \{1 + 4\kappa^2 y^{+2} [1 - \exp(-y^+/A^+)]^2\}^{1/2}}$ $\kappa = 0.4 \quad A^+ = 26$	All y^+	van Driest [14]
$u^+ = 2.5 \ln(1 + 0.4y^+)$ $+ 7.8[1 - \exp(-y^+/11)$ $- (y^+/11) \exp(-0.33y^+)]$	All y^+	Reichardt [15]

$$\frac{du^+}{dy^+} = \frac{1}{1 + n^2 u^+ y^+ [1 - \exp(-n^2 u^+ y^+)]}$$

$$n = 0.124$$

$$u^+ = 2.78 \ln y^+ + 3.8$$

$$0 < y^+ < 26$$

Deissler [16]

$$y^+ = u^+ + A[\exp Bu^+ - 1 - Bu^+ - \frac{1}{2}(Bu^+)^2 - \frac{1}{6}(Bu^+)^3 - \frac{1}{24}(Bu^+)^4]$$

(last term in u^{+4} may be omitted)

All y^+

$$A = 0.1108$$

$$B = 0.4$$

Spalding [17]

Source: After Ref. 10.

the wall shear stress scales as $\mu U_\infty / y_{\text{VSL}}$

$$y_{\text{VSL}}^+ \text{ at transition} \rightarrow y_{\text{VSL}}^+ = \frac{y_{\text{VSL}}}{\nu} \left(\frac{\tau_0}{\rho} \right)^{1/2} = \left(\frac{y_{\text{VSL}} U_\infty}{\nu} \right)^{1/2} \rightarrow y_{\text{VSL}}^+ \sim 10$$

the $y_{\text{VSL}}^+ \sim 10$ theory presented *first eddy* that forms immediately after the laminar shear layer

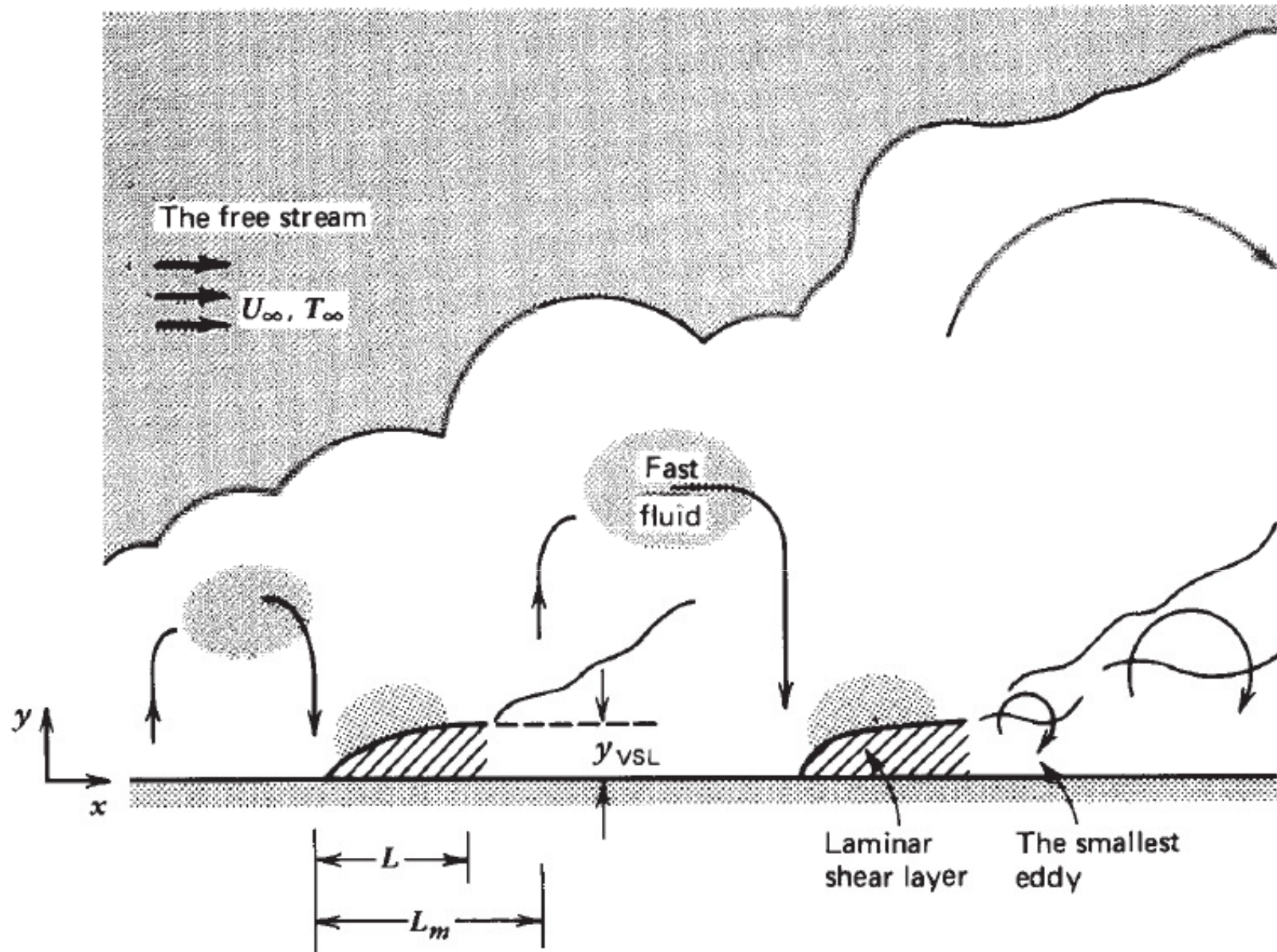


Figure 7.5 Formation of the viscous sublayer as the time-averaged superposition of laminar shear layers with *local Reynolds numbers* no greater than $\sim 10^2$.

$$\left. \begin{aligned} \epsilon_M &= \kappa^2 y^2 \left| \frac{\partial \bar{u}}{\partial y} \right| \\ \left(1 + \frac{\epsilon_M}{\nu}\right) \frac{du^+}{dy^+} &= 1 \end{aligned} \right\} \left[1 + \kappa^2 (y^+)^2 \frac{du^+}{dy^+}\right] \frac{du^+}{dy^+} = 1 \longrightarrow \kappa u^+ = \frac{\cos \alpha - 1}{\sin \alpha} + \ln \left[\tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right]$$

$$\alpha = \arctan(2\kappa y^+)$$

$$u^+ \rightarrow y^+ \quad \text{as} \quad y^+ \rightarrow 0$$

$$u^+ \rightarrow \frac{1}{\kappa} \ln y^+ + \frac{2 \ln 2 + \ln \kappa - 1}{\kappa} \quad \text{as} \quad y^+ \rightarrow \infty$$

$$\text{if } \kappa \cong 0.4, \longrightarrow u^+ = 2.5 \ln y^+ - 1.325$$

WALL FRICTION IN BOUNDARY LAYER FLOW

$$C_{f,x} = \frac{\tau_0}{\frac{1}{2} \rho U_\infty^2} \quad \frac{U_\infty}{(\tau_0/\rho)^{1/2}} \cong f_u \left[\frac{\delta}{\nu} \left(\frac{\tau_0}{\rho} \right)^{1/2} \right]$$

$$u^+ \cong f_u(y^+)$$

$$\frac{d\bar{P}}{dx} = 0 ;$$

$$\frac{d}{dx} \int_0^\infty \bar{u}(U_\infty - \bar{u}) dy = \frac{\tau_0}{\rho}$$

Prandtl's one-seventh power law as the fit for the $u^+(y^+)$ data $\rightarrow f_u = 8.7(y^+)^{1/7}$

$$\frac{\tau_0}{\rho U_\infty^2} = 0.0225 \left(\frac{U_\infty \delta}{\nu} \right)^{-1/4} \quad \frac{\delta}{x} = 0.37 \left(\frac{U_\infty x}{\nu} \right)^{-1/5} \quad \delta = 8\delta^* = \frac{72}{7} \theta$$

$$\frac{\tau_0}{\rho U_\infty^2} = \frac{1}{2} C_{f,x} = 0.0296 \left(\frac{U_\infty x}{\nu} \right)^{-1/5}$$

$$\frac{\tau_{0-x}}{\rho U_\infty^2} = \frac{1}{2} C_{f,0-x} = 0.037 \left(\frac{U_\infty x}{\nu} \right)^{-1/5}$$

Schultz-Grunow's empirical correlation [18]

$$C_{f,x} = 0.37 \left[\log_{10} \left(\frac{U_\infty x}{\nu} \right) \right]^{-2.584}$$

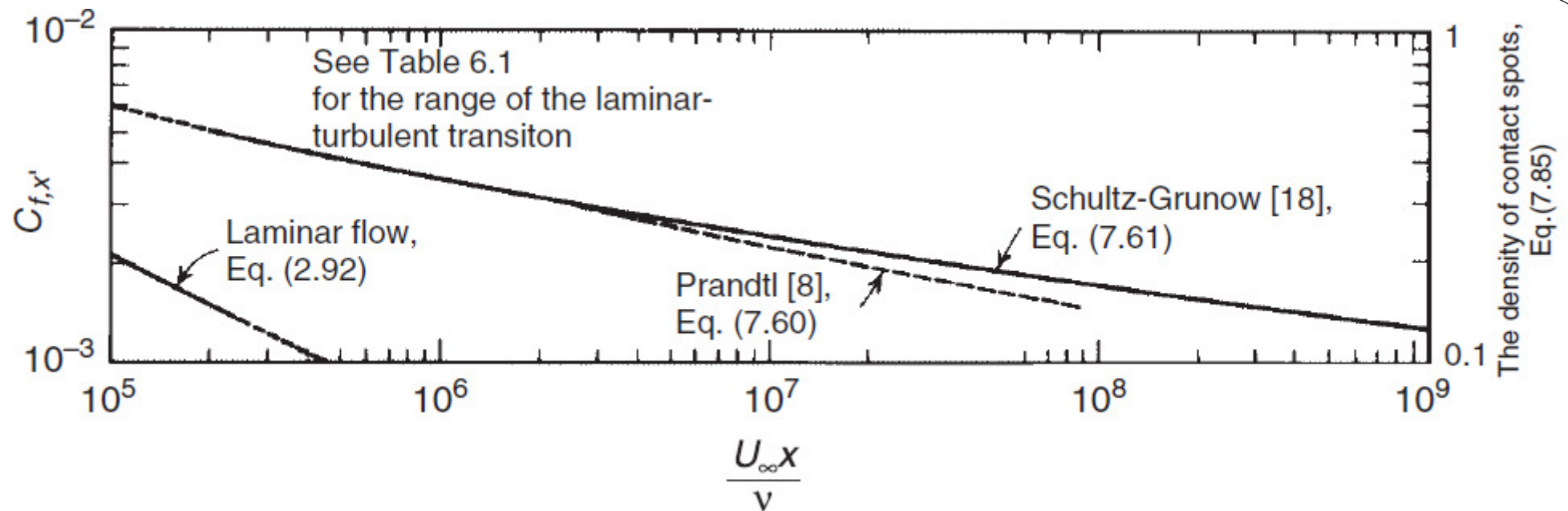


Figure 7.6 Local skin friction coefficient for turbulent boundary layer flow over a plane wall.

HEAT TRANSFER IN BOUNDARY LAYER FLOW

the apparent heat flux q''_{app} does not depend on y

$$(\alpha + \epsilon_H) \frac{\partial \bar{T}}{\partial y} = \left[(\alpha + \epsilon_H) \frac{\partial \bar{T}}{\partial y} \right]_{y=0} \rightarrow (\alpha + \epsilon_H) \frac{\partial \bar{T}}{\partial y} = \frac{-q''_0}{\rho c_P}$$

$$\frac{\rho c_P u_*}{-q''_0} \frac{\partial}{\partial y^+} (\bar{T} - T_0) = \frac{1}{\alpha/\nu + \epsilon_H/\nu}$$

$$T^+(x^+, y^+) = (T_0 - \bar{T}) \frac{\rho c_p u_*}{q_0''} \quad \text{the integral} \rightarrow T^+ = \int_0^{y^+} \frac{dy^+}{1/\text{Pr} + (1/\text{Pr}_t)(\epsilon_M/\nu)}$$

the *turbulent Prandtl number* ($\text{Pr}_t = \epsilon_M/\epsilon_H$)

assumptions regarding Pr , Pr_t , and ϵ_M/ν

in the fully turbulent region $\frac{\epsilon_M}{\nu} = \frac{dy^+}{du^+} = \kappa y^+$

$$T^+ = \int_0^{y_{\text{CSL}}^+} \frac{dy^+}{\frac{1}{\text{Pr}} + \left(\begin{array}{c} \text{negligible} \\ \text{term} \end{array} \right)} + \int_{y_{\text{CSL}}^+}^{y^+} \frac{dy^+}{\left(\begin{array}{c} \text{negligible} \\ \text{term} \end{array} \right) + \frac{1}{\text{Pr}_t} \frac{\epsilon_M}{\nu}}$$

y_{CSL}^+ is the dimensionless thickness of a *conduction sublayer* (CSL)
the molecular mechanism outweighs the eddy transport of heat.

$$T^+ = \begin{cases} \text{Pr } y^+, & y^+ < y_{\text{CSL}}^+ \\ \text{Pr } y_{\text{CSL}}^+ + \frac{\text{Pr}_t}{\kappa} \ln \frac{y^+}{y_{\text{CSL}}^+}, & y^+ > y_{\text{CSL}}^+ \end{cases} \quad \begin{array}{l} \text{Pr}_t \cong 0.9, \\ \kappa \cong 0.41, \\ y_{\text{CSL}}^+ \cong 13.2 \end{array}$$

$y^+ > y_{\text{CSL}}^+$ portion of the T^+ profile $\longrightarrow T^+ = 2.195 \ln y^+ + 13.2\text{Pr} - 5.66$

Prandtl number of the fluid is in the range 0.5–5.

Setting $\bar{T} = T_\infty$ at $y = \delta$

$$T^+(x^+, y^+) = (T_0 - \bar{T}) \frac{\rho c_P u_*}{q_0''}$$

$$T^+ = \begin{cases} \text{Pr } y^+, & y^+ < y_{\text{CSL}}^+ \\ \text{Pr } y_{\text{CSL}}^+ + \frac{\text{Pr}_t}{\kappa} \ln \frac{y^+}{y_{\text{CSL}}^+}, & y^+ > y_{\text{CSL}}^+ \end{cases}$$

$$\left. \begin{array}{l} \rho c_P u_* \frac{T_0 - T_\infty}{q_0''} = \text{Pr } y_{\text{CSL}}^+ + \frac{\text{Pr}_t}{\kappa} \ln \frac{\delta u_* / \nu}{y_{\text{CSL}}^+} \\ h = q_0'' / (T_0 - T_\infty) \end{array} \right\}$$

$$\left. \begin{array}{l} \frac{U_\infty}{u_*} = \frac{1}{\kappa} \ln \frac{\delta u_*}{\nu} + B \\ \frac{U_\infty}{u_*} = \left(\frac{2}{C_{f,x}} \right)^{1/2} \end{array} \right\}$$

$$\frac{h}{\rho c_P U_\infty} = \frac{\frac{1}{2} C_{f,x}}{\text{Pr}_t + \left(\frac{1}{2} C_{f,x} \right)^{1/2} [\text{Pr } y_{\text{CSL}}^+ - B \text{Pr}_t - (\text{Pr}_t / \kappa) \ln y_{\text{CSL}}^+]}$$

$$\text{St}_x = \frac{h}{\rho c_P U_\infty} = \frac{\text{Nu}_x}{\text{Pe}_x} = \frac{\text{Nu}_x}{\text{Re}_x \text{Pr}} \quad \text{local Stanton number}$$

$$B \cong 5.1$$

$$\text{St}_x = \frac{\frac{1}{2} C_{f,x}}{0.9 + \left(\frac{1}{2} C_{f,x} \right)^{1/2} (13.2\text{Pr} - 10.25)}$$